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DYMAMICS OF STRUCTURES

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2005

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CHAPTER (1)

OVERVIEW OF STRUCTURAL DYNAMICS

1.1 OBJECTIVE OF STRUCTURAL DYNAMICS ANALYSIS

- Present methods for analyzing stresses and deflections in a certain structure when subjected to an arbitrary dynamic loading.
- Or extension of standard methods of static analysis to account for dynamic loads. So static load will be a special case of dynamic load.
- Dynamic means time varying i.e. dynamic load is any load of which the magnitude and direction or position varies with time. Then the external forces called dynamic or excitation forces produce displacements of the system called dynamic response.

There are two basically different approaches for evaluating structural response to dynamic loads:

(a) *Deterministic approach;* if the time variation of the load is fully known and even though it may be irregular in character (prescribed dynamic loading).

(b) *Non-deterministic approach*; if the time variation of the load is not completely known (because it never repeats itself) but can be defined in a statistical sense. For example forces produced by wind flow or earthquakes (Random dynamic loading).

Response is expressed basically in terms of the displacement of the structure. So in (a) we get displacement-time history corresponding to the prescribed load history. Then other aspects of response: stresses, strains, internal forces are obtained as a secondary phase from the other established displacement.

On the other hand in (b) the load is defined statistically and so is the displacement, i.e. time variation of displacement is not determined. So, stresses etc. are determined directly by independent nondeterministic analysis rather than from the displacement results.

1.2 TYPES OF PRESCRIBED LOADINGS

The type of response of a structure depends on the nature of the loads applied. The loads and the response resulting from them can be Periodic or Transient (Non-Periodic).

a. periodic loads: repetitive loads which exhibits the same time variation successively for a large number of cycles. The simplest harmonic loading is the sinusoidal variation shown in Figure 1.1.a, such loading can be produced due to unbalanced mass effects in rotating machinery.

Harmonic force may be presented by the projection of a rotating vector of a centrifugal force in the vertical or horizontal direction,

$$\mathbf{P_v = P_o \sin \Omega\, t}$$

$$\mathbf{P_H = P_o \cos \Omega\, t}$$

Where, Ω is circular frequency and P_o is the amplitude of oscillation.

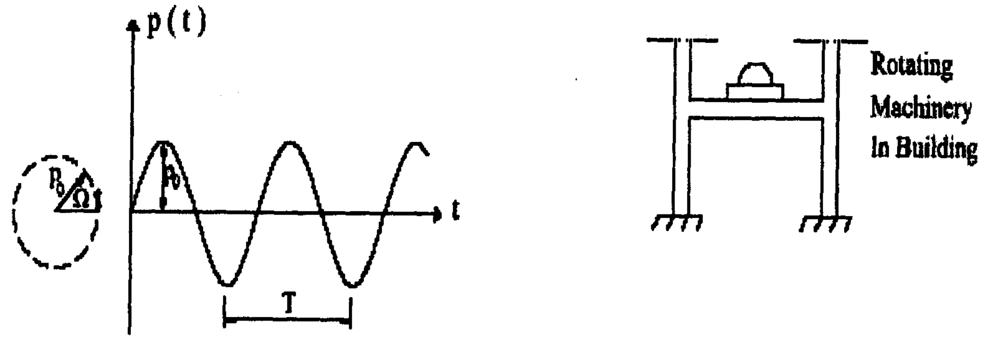


Figure 1.1.a Simple harmonic loading.

The period measured in seconds is

$$T = \frac{2\pi}{\Omega}$$

in one period T one oscillation is completed, i.e., $\Omega T = 2\pi$. The frequency measured in cycles per seconds (Hertz or HZ) is

$$F = \frac{1}{T} = \frac{\Omega}{2\pi} \qquad \text{HZ}$$

Figure 1.1.b shows an example of *complex harmonic* generated by a propeller at the stern of a ship or unbalance in resiprocating machines. Any complex harmonic can be represented as the sum of a series of simple harmonic components by Fourier analysis. Therefore, knowledge of the Harmonic case facilitates the treatment of more complicated types of excitation.

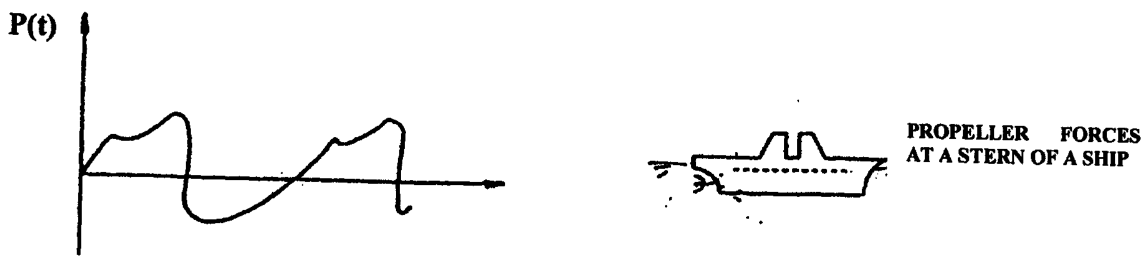


Figure 1.1.b Complex harmonic loading.

b. ***Transient loading:*** it is characterized by nonperodic time history of a limited duration. It may be either short-duration impulsive loading or long-duration general forms. For impulsive loads (Produced by hammer blows, collisions, blasts etc.), special simple forms of analysis may be employed. On the otherhand, a general long-duration loading (generated by earthquakes) can be treated by completely general dynamic analysis procedures.

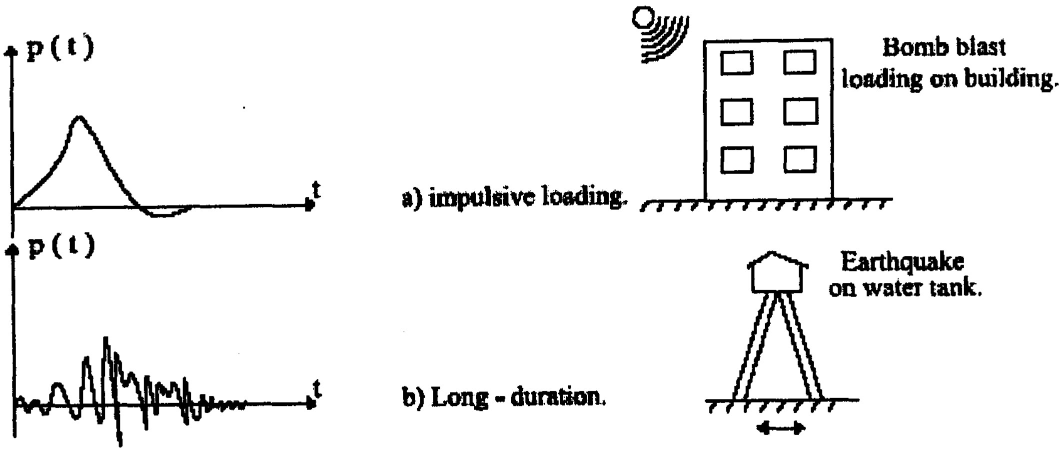


Figure 1.2 **Nonperiodic loading.**

1.3. ESSENTIAL CHARACTERISTIC OF A DYNAMIC PROBLEM

There are two important distinctions between static and dynamic problems:

a. Dynamic loading is ***time-varying.*** So, it does not have a single solution but it has solutions corresponding to all times of interest in the response history. Thus dynamic analysis is more complex and time consuming than static analysis.

b. ***Inertia Forces*** which resist acceleration of the structure is the most characteristic of a structural dynamic problem. So, for a static load, internal forces and displacements are obtained by established principles of forces equilibrium. On the other hand, a dynamic load produces not only displacements but also accelerations which result in inertia forces to resist the accelerations. Thus the internal forces of the structure (e.g. the beam in Figure 1.3) must equilibrate not only the externally applied force, but also the inertia forces resulting from the acceleration of the structure.

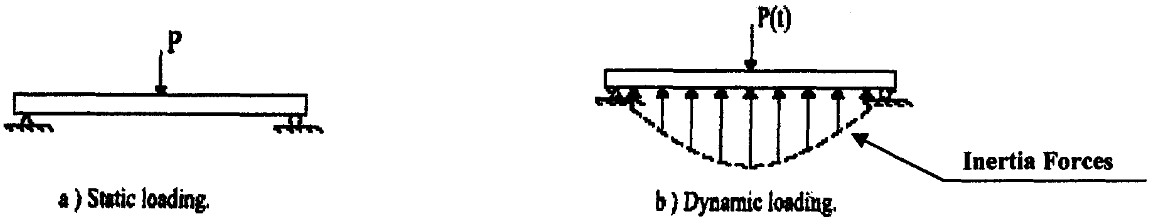


Figure 1.3 **Basic Difference between static and dynamic Loads.**

1.4 METHODS OF DISCRETIZATION

a. Lumped-mass procedure

The dynamic analysis of any system is complicated by the fact that the inertia forces result from the structural displacements which in turn are influenced by the magnitude of the inertia force. This closed cycle of cause and effect can be attacked directly only by formulating the problem in differential equations. Furthermore, because the mass of the structure (e.g. the beam in Figure 1.3)

is continuously distributed along its length, the displacements and acceleration must be defined along its axis. In this case the analysis must be formulated in terms of partial differential equations because the position along the span as well as the time must be taken as independent variables.

On the other hand, if the mass of the beam were concentrated at discrete points or lumps, the inertia forces could be developed only at these points and the analysis would be greatly simplified (Figure 1.4)

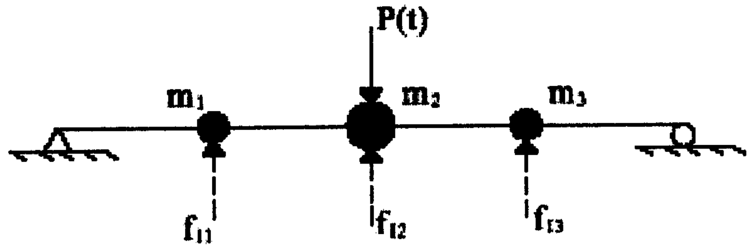


Figure 1.4 Lumped mass idealization of a simple beam.

The number of displacement components which must be considered in order to represent the effects of all significant inertia forces may be termed the number of dynamic degrees of freedom (No. DOF). For example the beam in Figure 1.4 has 3 DOF, for three dimensional case, the beam will have 18 DOF (6 fore each mass). So, a system with continuously distributed mass has an infinite number of degrees of freedom.

The systems considered in dynamics are the same as those met in statics, i.e. buildings, bridges, towers, dams, water tanks, silos etc. For the dynamic analysis of a system, the suitable mathematical model must be chosen. Examples of distributed and lumped mass models are shown in Figure 1.5. As the number of concentrated masses increases, the lumped mass model converges to the distributed model. In rigid bodies (e.g. silos), mass moment of inertia is considered as well as mass.

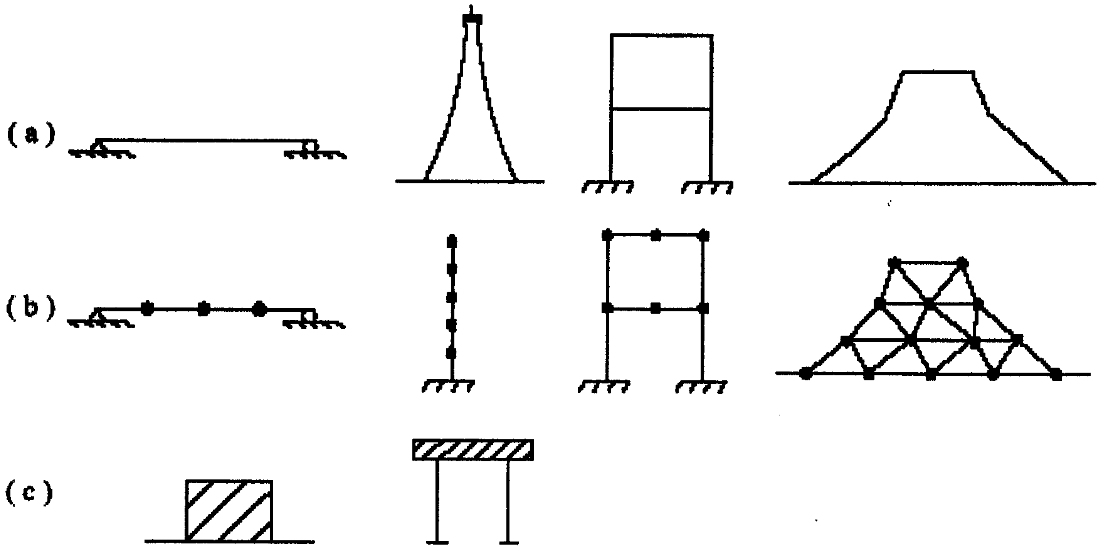


Figure 1.5 (a) Distributed models, (b) Lumped mass models and (c) Rigid bodies.

b. Generalized Displacements

In cases where the mass of the system is quite uniformly distributed throughout, the degrees of freedom may be limited by expressing the deflected shape as the sum of series of specified displacement patterns (displacement coordinate of the structure).

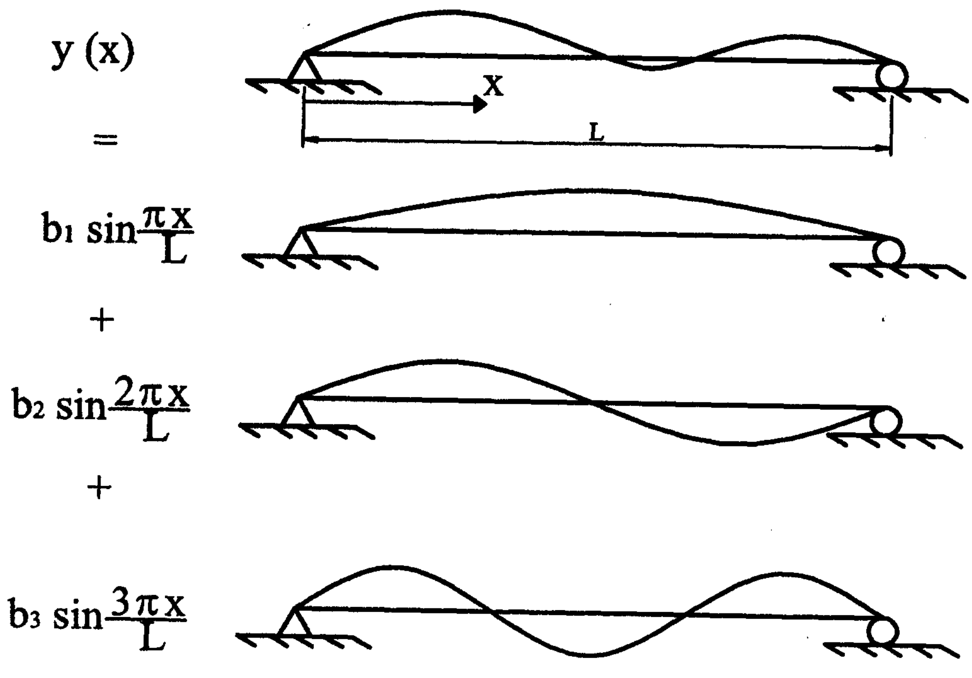
<u>Ex:</u> For the beam

$$y(\chi) = \sum_{n=1}^{\infty} b_n \quad \sin \frac{n\,\pi\,\chi}{\ell}$$

In general any arbitrary shape compatible with the prescribed support conditions can be represented by an infinite series of such sine–wave components. The amplitude of the sine wave shapes may be considered to be the coordinates of the system, and the infinite number of degrees of freedom are represented by the infinite number of terms in the series. A good approximation can be achieved by truncating the series; thus a 3DOF approximation would contain only 3 terms in the series, etc. Generally, any shape $\psi_n(\chi)$ which are compatible with the prescribed geometric-support conditions and which maintain the necessary continuity of internal displacements may be assumed.

$$\therefore y(\chi) = \sum_{n=1}^{N} Z_n \,\psi_n\,(\chi)$$

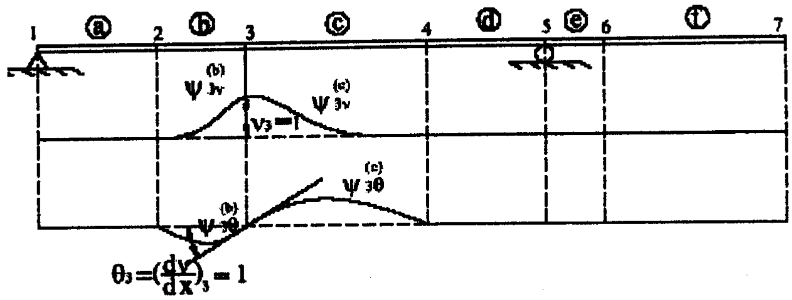
For any assumed set of functions $\psi_n\,(\chi)$, the resulting shape of the structures depends upon the amplitudes Z_n (generalized coordinates). The number of assumed shape patterns (functions) ≡ number of DOF in this idealization. Generally, the shape function method of idealization results in better accuracy than the lumped mass approach but requires more computional efforts, however.



Sine-series representation of simple beam deflection.

c. The Finite Element Method
In this method, the displacements of the system are expressed in terms of a finite number of descrete displacement coordinate, which combines certain feachers of both lumped mass and the generalized coordinates.

Ex: The beam in the Figure is divided into an appropriate number of segments or elements (a, b, c, d), which are interconnected at the nodal points (1, 2, 3, 4 and 5). The displacements of the nodal points then become the generalized coordinates of the structure.



Typical finite-element beam coordinates.

The deflection of the complete structure can now be expressed in terms of these generalized coordinates by means of an appropriate set of assumed displacement functions (interpolation functions). The interpolation functions could be any curve which is internally (in each element) continuous and which satisfies the geometric displacement conditions imposed by the nodal displacements. Of course, the computational efforts are high and can be done only by using the digital computers. Finite element people claim that the FEM provides the most efficient procedure for expressing the displacements of arbitrary structural configurations by means of a descrete set of coordinates.

1.5 FORMULATION OF THE EQUATION OF MOTION

The mathematical expressions defining the dynamic displacement of a structure are called the Equations of Motion and the solution of these equations of motion provides the required displacement histories. The formulation of the equation of motion of a dynamic system is the most important (and sometimes the most difficult) phase of the entire analysis procedure. Different methods will be employed for the formation of the these equations, each having advantages in the study of special classes of problems.

a. Direct Equilibrium Using d' Alembert Principle
The equations of motion of any dynamic system represent expressions of Newton's 2^{nd} law of motion, which states that "the rate of change of *momentum* of any mass **m** is equal to the force acting on it".

$$\mathbf{F(t)} = \frac{\mathbf{d}}{\mathbf{dt}}\left(\mathbf{m}\frac{\mathbf{dy}}{\mathbf{dt}}\right) \tag{1.1}$$

where, **F(t)** is the force vector acting on the mass **m**, **y(t)** is the position vector of the mass **m**
If the mass does not vary with time, Equation 1.1, becomes

$$\mathbf{F(t) = m\frac{d^2y}{dt^2} = m\ddot{y}} \tag{1.2}$$

i.e. the forces = mass × acceleration.

The term $\mathbf{m\ddot{y}}$ is the ***inertia force*** resisting the acceleration of the mass.

The concept that the mass develops an inertia force proportional to the acceleration and opposing it is known as ***d' Alembert principle.*** It is a very convenient device in structural dynamics because it permits the equations of motion to be expressed as equations of dynamic equilibrium.

The force **F(t)** may include many types of force acting on the mass: i) elastic constraint which oppose displacement $\mathbf{F_s=k.y}$, ii) viscous force which resist velocities $\mathbf{F_D=c\dot{y}}$, iii) and independent defined external loads **P(t)**.

Thus if an inertia force which resist acceleration is introduced, the expression of the equation of motion is merely an experession of equilibrium of all the forces acting on the mass. In many simple problems, the most direct and convenient way of formulating the equations of motion is by means of such direct equilibrium.

b. Principle of Virtual Work

If the structure is reasonably complex, the direct equilibrium of all the forces acting on the system may be difficult. This is because the equilibrium relationships of the forces in such case may be obsecure. In this case the principle of virtual displacement can be used to formulate the equations of motion as a substitute for the equilibrium relationships.

This principle can be expressed as follows: If a system which is in equilibrium under a set of forces is subjected to a virtual displacement (any displacement compatible with the system constraints), the total work done by the forces will be zero.

Steps are:

- Identify all forces acting on the masses of the system including inertia forces (according to d'Alembert principle).
- Introduce virtual displacements corresponding to each degree of freedom
- Equate the work done to zero

c. Hamilton's Principles

Using the variational form of the energy quantities, the equation of motion can be obtained. Hamilton's principle may be expressed as:

$$\mathbf{\int_{t_1}^{t_2} \delta(T - V)dt + \int_{t_1}^{t_2} \delta W_{nc}dt = 0} \tag{1.3}$$

where **T** = total kinetic energy of system

V = Potential energy of system, including both strain energy and potential of any conservative external loads.

$\mathbf{W_{nc}}$ = Work done by nonconservative forces acting on system, including damping and arbitrary external loads.

δ = Variations taken during indicated time interval

"The variation of the kinetic and potential energy plus the variation of the work done by the nonconservative forces considered during any time interval t_1 to t_2 must equal zero". The application of this principle leads directly to the equations of motion for any given system.

CHAPTER (2)

FORMULATION OF THE EQUATION OF MOTION

2.1 ONE DEGREE OF FREEDOM STRUCTURES

Figures 2.1a and 2.1d show some examples of structures which may be represented for dynamic analysis as one degree of freedom system; that is, structures modeled as systems with a single displacement coordinate. These single DOF systems may be described conveniently by the mathematical models shown in Figure 2.1b and 2.1e respectively. Figure 2.1c and 2.1f show the free body diagrams (sketch of the body isolated from other bodies and all the forces external to the body are shown).

The forces acting on the mass **m** in the direction of the displacement degree of freedom include: the applied load **P(t)** and the three forces resulting from the motion; elastic spring force F_S, damping force F_D and the inertia force F_I.

The rollers constrain the rigid block so it can move only in simple translation; thus single displacement coordinate **y** completely defines its position.

The forces on the mass **m** are descried as follows:

a. ***External forces*** are the direct cause of dynamic response. They may be any of the types described in subchapter 1.2.

b. ***Elastic (restoring) forces*** always tend to return the structure into the position of equilibrium. They can be expressed, as in statics, by means of stiffness, **k**, which is defined as the staic force requird to produce a unit displacement. Table 2.1 gives the stiffness constants for a few structural elements. The spring (restoring) force generated in a structure by displacement **y (t) = y** is

$$F_s = k.y$$

c. ***Damping forces*** oppose the motion through energy dissipation. This dissipation occurs due to imperfect elasticity of the material manifested by a hysteric loop, friction between elements of the structure, resistance of fluid surrounding the structure such as air, propagation of waves through soil, etc. The dissipated energy cannot be recovered.

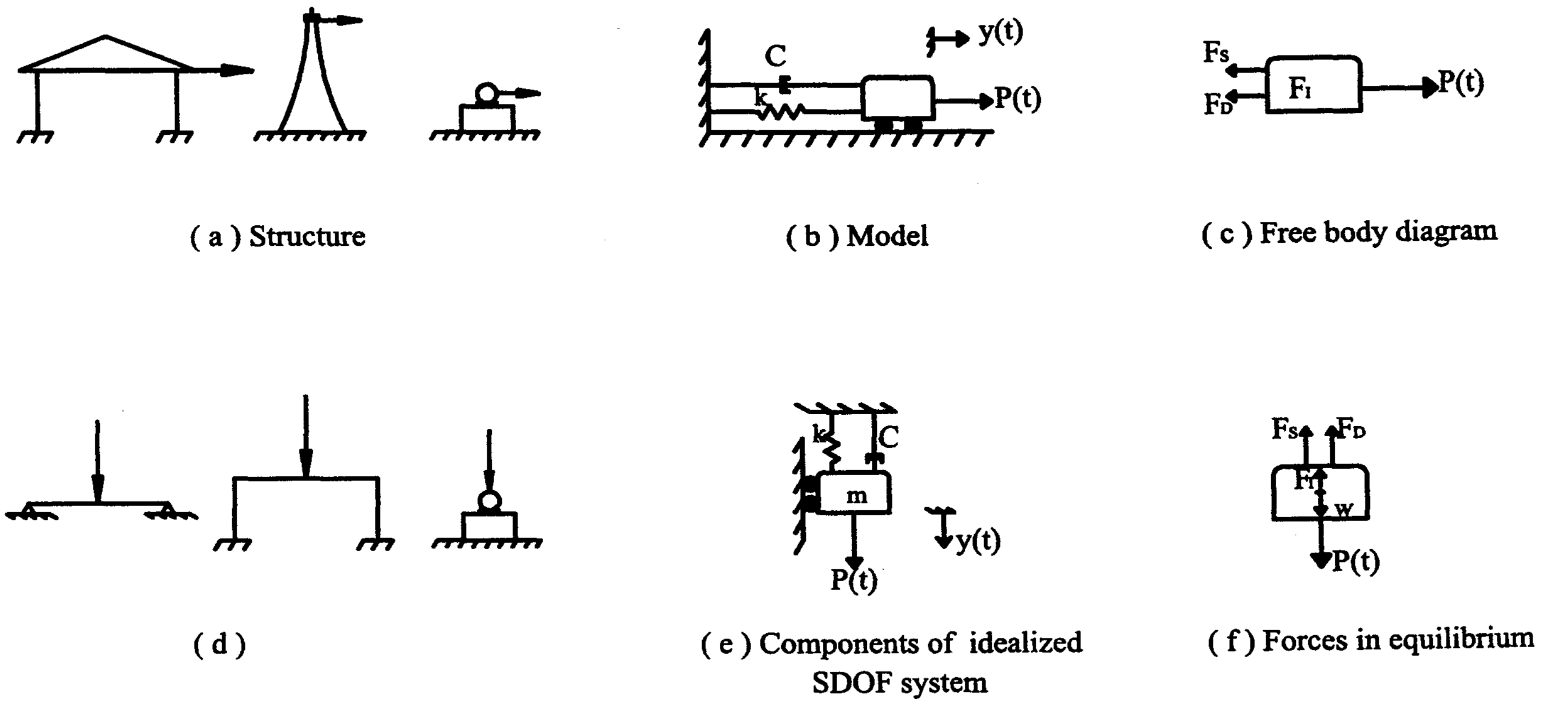
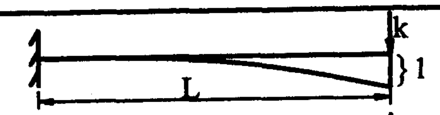
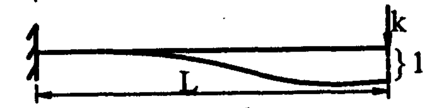
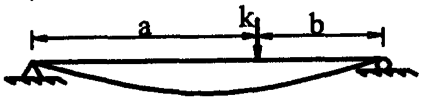
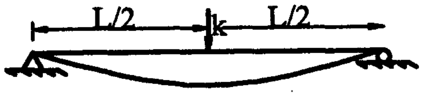
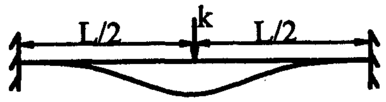
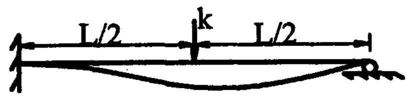
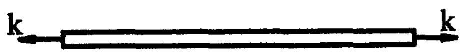
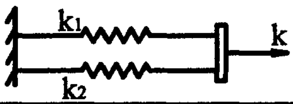


Figure 2.1 Structures modeled as one D.O.F systems.

Table 2.1
Stiffness Constants

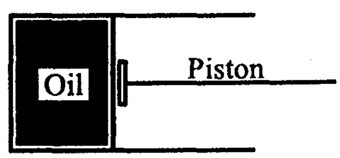
No.	Structure	k =	Comment
1		$\frac{3EI}{L^3}$	Fixed - Pinned
2		$\frac{12EI}{L^3}$	Fixed - Fixed
3		$\frac{3EIL}{a^2b^2}$	Pinned - Pinned
4		$\frac{48EI}{L^3}$	Pinned - Pinned
5		$\frac{192EI}{L^3}$	Fixed - Fixed
6		$\frac{768EI}{7L^3}$	Fixed - Pinned
7		$\frac{EA}{L}$	Axial
8		$\frac{GJ}{L}$	Torsion
9		$\frac{1}{1/k_1 + 1/k_2}$	Springs in Series
10		$k_1 + k_2$	Parallel Springs

I = moment of inertia of cross-sectional area
A= cross-sectional area
J= torsional constant of cross section
L= length of element

The exact cause and nature of damping (energy dissipation) are often difficult to describe accurately. The most commonly, used type, of damping is the ***viscous*** damping. It is modeled by a dashpot consisting of a cylinder filled with a viscous fluid in which a piston is free to move. The damping force of a dashpot is proportional to the magnitude of the velocity and opposite to the direction of the motion and is

$$F_D = c\frac{dy(t)}{dt} = c\dot{y}$$

Dashpot



in which **c** is the constant of viscous damping defined as the force associated with unit velocity. The viscous damping well describes the resistance of viscous fluids such as air or water. Other models of damping are also used, however.

d. ***Inertia forces*** are defined as the product of the mass, **m**, and the acceleration, $\ddot{y}$

$$F_I = m\frac{d^2y(t)}{dt^2} = m\ddot{y}$$

If an angular displacement of a rigid body is involved, the inertia force is equal to the mass moment of inertia, I_o multiplied by angular acceleration, $\ddot{\psi}$,

$$m_I = I_o.\ddot{\psi}$$

The mass and mass moment of inertia of a uniform rod and of a uniform plate are summarized in the following figure.

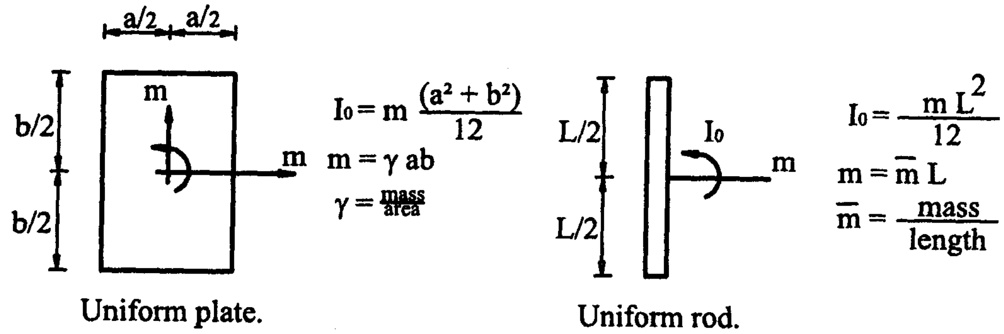


Figure 2.2 Rigid body mass and mass moment of inertia.

2.2 METHODS OF FORMULATION

Using any of the three procedures discussed in chapter (1), the equation of motion can be formulated as follows:

2.2.1 Direct Equilibrium

Equilibrium of all forces acting on the mass m in the direction of the displacement degree of freedom results in the equation of motion directly, as follows:

$$F_I + F_D + F_s = P(t) \tag{2.1}$$

or $$m\ddot{y} + c\dot{y} + ky = P(t) \tag{2.2}$$

By d'Alembert principle   viscous damping   elastic force

Equation 2.2 is known as the equation of motion of single degree of freedom system (SDOF).

2.2.2 Virtual-Work Analysis

If the mass is given virtual displacement δy, the forces acting on the mass will each do work. The total work done by the system can then be written as:

$$-F_I\,\delta y - F_D\,\delta y - F_s\,\delta y + P(t)\,\delta y = 0 \tag{2.3}$$

or $$[-m\ddot{y} - c\dot{y} - ky + P(t)]\delta y = 0 \tag{2.4}$$

since $\delta y \neq 0$

$$\therefore m\ddot{y} + c\dot{y} + ky = P(t) \tag{2.5}$$

2.2.3 Application of the Hamilton's Principle

The kinetic energy of the system $T = \frac{1}{2}m\dot{y}^2$ (2.6)

The strain energy of the spring $V = U = \frac{1}{2}ky^2$ (2.7)

The nonconservative forces are F_D **and** $P(t)$, the variation of the work done by these forces is:

$$\delta W_{nc} = P(t)\delta y - c\dot{y}\delta y \tag{2.8}$$

Apply Hamilton principle

$$\int_{t_1}^{t_2} \delta(T - V)dt + \int_{t_1}^{t_2} \delta W_{nc}dt = 0$$

or $$\int_{t_1}^{t_2} \delta\left[\frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2\right] + \int_{t_1}^{t_2} \delta W_{nc}dt = 0$$

or $$\int_{t_1}^{t_2} [m\dot{y}\delta\dot{y} - ky\delta y] + \int_{t_1}^{t_2} [P(t)\delta(- c\dot{y}\delta y]dt = 0$$

or $$\int_{t_1}^{t_2} [m\dot{y}\delta\dot{y} - c\dot{y}\delta y - ky\delta y + P(t)\delta y]dt = 0 \tag{2.9}$$

$$m\ddot{\bar{y}} + c\dot{\bar{y}} + k\Delta_{st} + k\bar{y} = P(t) + w$$

or $$m\ddot{\bar{y}} + c\dot{\bar{y}} + k\bar{y} = P(t) \tag{2.15}$$

That is to say that the equation of motion expressed with references to the static equilibrium of the system is not affected by gravity forces. The displacement which is determined will be ***the dynamic response***. Therefore the total deflections, stresses, etc. can be obtained only by adding the appropriate static quantities to the results of the dynamic analysis (Figure 2.4).

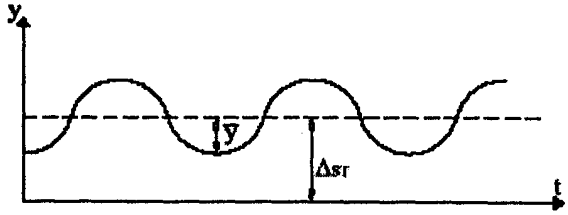


Figure 2.4 Total displacement = static displacement +dynamic displacement.

2.4 INFLUENCE OF SUPPORT EXCITATION

Dynamic stresses and deflections may be induced in a structure by motions of its support points. Example of such excitation is the motions of a building foundation caused by an earthquake. A simplified model of the earthquake excitation problem is shown in Figure 2.5 in which the horizontal ground motion caused by the earthquake is indicated by the displacement $\mathbf{u_g}$ of the structure base relative to the fixed reference axis. The equilibrium forces of this system is

$$F_I + F_D + F_s = 0 \tag{2.16}$$

Since the total (absolute) displacement of mass from the reference axis is

$$u_t = u + u_g$$

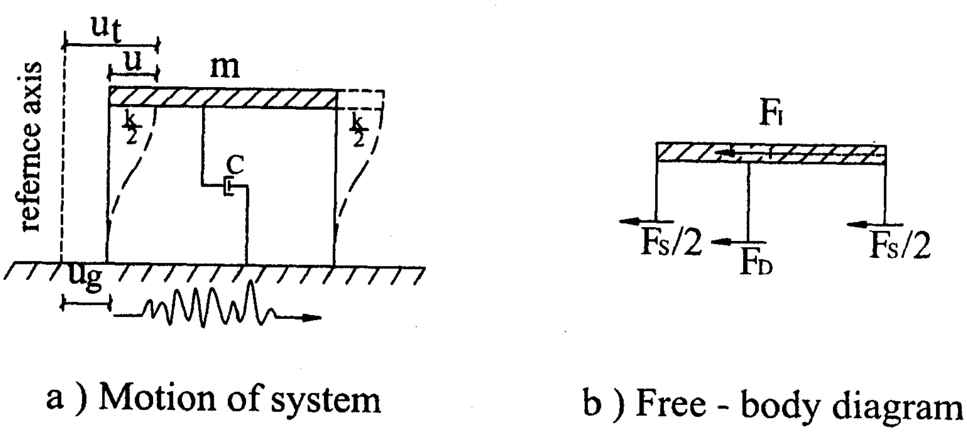


Figure 2.5 Influence of support excitation on SDOF equilibrium.

where, $\mathbf{u}$ = relative displacement of mass. Then, the inertia, damping, and elastic forces are:

$$\mathbf{F_I = m\ddot{u}_t = m\ddot{u} + m\ddot{u}_g}$$
$$\mathbf{F_D = c\dot{u}} \quad , \quad \mathbf{F_S = ku} \tag{2.17}$$

Substituting Equation 2.17 into Equation 2.16.

$$\mathbf{m\ddot{u} + m\ddot{u}_g + c\dot{u} + ku = 0}$$
or
$$\mathbf{m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g} \tag{2.18}$$

In this equation, the term $\mathbf{-m\ddot{u}_g}$ represents the effective support excitation loading. The (-ve) sign indicates that the effective force oppose the direction of the ground acceleration.

2.5. GENERALIZED SDOF SYSTEM

Two classes of generalized SDOF structures:

a. Assemblages of rigid bodies in which the elastic deformations are limited to localized spring elements.
b. Systems having distributed elasticity in which the deformations may be continuous throughout the structure.

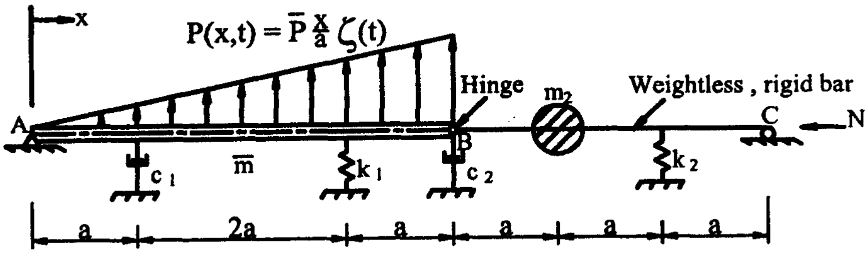
In both cases, the structure is forced to behave like a SDOF system by assuming that displacements of only a single form or shape are permitted.

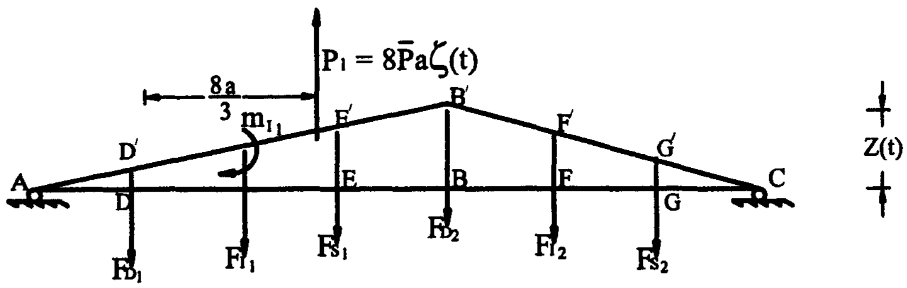
For case (a) the limitation to a single displacement shape is a consequence of the assemblage configuration; for case (b) the SDOF shape restriction is merely an assumption since it has an infinite variety of displacements to take place.

2.5.1 Rigid Body Assemblage

In formulating the Equation of motion of the rigid body assemblage the elastic and damping forces developed during the SDOF displacements can be expressed easily in terms of the displacement and velocity amplitudes, respectively. On the other hand, the mass of rigid bodies need not be localized, and distributed inertia forces generally will result from the assumed accelerations. However, for the purpose of dynamic analysis, it usually is most effective to treat the rigid-body inertia forces as though the mass and the mass moment of inertia were concentrated at the center of the mass. Similarly, the distributed external loads are represented by their resultant. The mass and the mass moment of inertia of a uniform rod and a uniform plate are summarized in Figure 2.2.

Example:





For the form of displacement which may take place in this SDOF structure, the hinge motion **Z(t)** may be taken as the basic quantity and all other displacements expressed in terms of it. Each resisting force can be expressed in terms of Z or its time derivatives as follows:

$$f_{S1} = \frac{3}{4}k_1 Z(t) \qquad f_{S2} = \frac{1}{3}k_2 Z(t)$$

$$f_{D1} = \frac{1}{4}c_1\dot{Z}(t) \qquad f_{D2} = c_2\dot{Z}(t) \tag{2.19}$$

$$f_{I1} = \bar{m}L\frac{\ddot{Z}}{2} = 2a\bar{m}\ddot{Z}(t) \qquad f_{I2} = \frac{2}{3}m_2\ddot{Z}(t)$$

$\ddot{Z}(t)$

$$m_{I1} = I_o\frac{1}{4a}\ddot{Z}(t) = \frac{\bar{m}L}{4a}\frac{L^2}{12} = \frac{4}{3}a^2\bar{m}\ddot{Z}(t)$$

$$P_1(t) = 8\bar{P}a\xi(t)$$

Applying virtual-work: by equating to zero all work done by these forces during an arbitrary displacement δZ;

$$-\frac{3}{4}k_1 Z(t)\left\{\frac{3}{4}\delta Z\right\} - \frac{k_2}{3}Z(t)\left\{\frac{\delta Z}{3}\right\} - \frac{c_1}{4}\dot{Z}(t)\left\{\frac{\delta Z}{4}\right\} - c_2\dot{Z}(t)\{\delta Z\} - 2a\bar{m}\ddot{Z}(t)\left\{\frac{\delta Z}{2}\right\}$$
$$-\frac{4}{3}a^2\bar{m}\ddot{Z}(t)\left\{\frac{\delta Z}{4a}\right\} - \frac{2}{3}m_2\ddot{Z}(t)\left\{\frac{2}{3}\delta Z\right\} + 8\bar{P}\xi(t)\left\{\frac{2}{3}\delta Z\right\} = 0$$

or

$$-\left[\left(\frac{4}{3}\bar{m}a + \frac{4}{9}m_2\right)\ddot{Z}(t) + \left(\frac{c_1}{16} + c_2\right)\dot{Z}(t) + \left(\frac{9}{16}k_1 + \frac{1}{9}k_2\right)Z(t) - \frac{16}{3}\bar{P}a\xi(t)\right]\delta Z = 0 \tag{2.20}$$

Since δZ is arbitrary $\therefore \delta Z \neq 0 \ \therefore [\quad] = 0$

$\therefore$ Equation of motions is

$$m^* \ddot{Z}(t) + c^* \dot{Z}(t) + k^* Z(t) = P^*(t) \tag{2.20a}$$

where

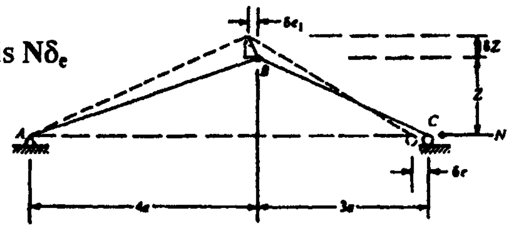
$$m^* = \frac{4}{3}\overline{m}a + \frac{4}{9}m_2 \qquad c^* = \frac{c_1}{16} + c_2$$

$$k^* = \frac{9}{16}k_1 + \frac{1}{9}k_2 \qquad P^* = \frac{16}{3}\overline{P}a\xi(t)$$

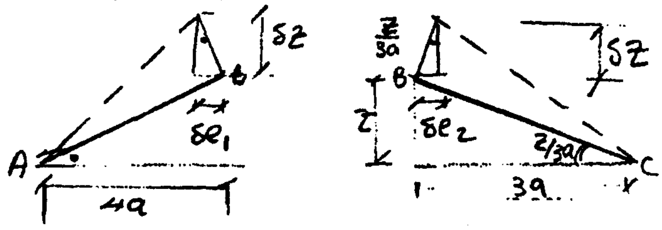
m*, c*, k* and p* are termed respectively generalized mass, damping, stiffness and load of this system

Effect of Normal Force N

Work done by N during virtual displacement δZ is $N\delta_e$



Displacement components in the direction of axial force



$\delta e = \delta e_1 + \delta e_2$

due to rotation of bar AB $= \dfrac{Z}{4a}\delta Z$

due to rotation of bar BC $= \dfrac{Z}{3a}\delta Z$

$$\therefore \delta W_N = N\delta e = \frac{7}{12}\frac{NZ}{a}\delta Z \tag{2.20b}$$

Introducing Equation 2.20b into Equation 2.20 and carrying out simplifying operations as before:

$$\therefore \overline{k}^* = k^* - \frac{7}{12}\frac{N}{a}$$

$$= \frac{9}{16}k_1 + \frac{1}{9}k_2 - \frac{7}{12}\frac{N}{a}$$

It is of interest to note that the condition of zero generalized stiffness represents a neutral stability or critical buckling condition in the system. Thus N_{cr} which cause bukling can be found from $\overline{k}^* = 0$

$$\therefore N_{cr} = \left(\frac{27}{28}k_1 + \frac{4}{21}k_2\right)a \tag{2.21}$$

Note: compressive axial forces tend to reduce the stiffness of a structure, while tensile axial forces cause a corresponding increase of stiffness.

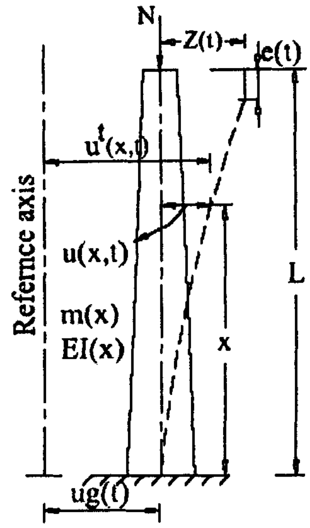
2.5.2. Distributed Flexibility

In the previous example, if the bars could deform in flexure, the system would have an infinite number of degrees of freedom. A simple SDOF analysis could still be made; however, if it were assumed that only a single deflection pattern could be developed including an appropriate flexural-deformation component.

Example

m(x) = mass/unit length
EI(x) = flexural stiffness
$\mathbf{u_g(t)}$ = E.Q. ground motion
N = axial load at top

To approximate the motion of this structure with a SDOF, it is necessary to assume that it may deflect only in a single Shape $\psi(x)$. Then amplitude of motion will be represented by the generalized coordinate; **Z(t)**



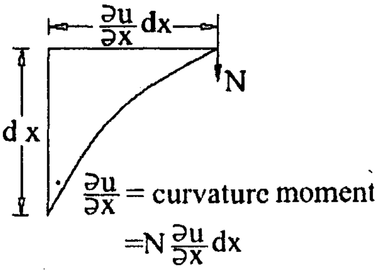
General structure treated as a SDOF system.

$$\therefore u(x,t) = \psi(x).Z(t) \tag{2.22}$$

Using Hamilton's principle to formulate the Equation of motion:

$$T = \frac{1}{2}\int_0^L m(x)\left[\dot{u}^t(x,t)\right]^2 dx$$

$$V = \frac{1}{2}\int_0^L EI\left[\frac{d^2u}{dx^2}(x,t)\right]^2 dx - \frac{N}{2}\int_0^L \left[\frac{du(x,t)}{dx}\right]^2 dx$$



nonconservative forces = 0

$$\therefore \int_{t_1}^{t_2} \delta(T-V)dt = 0$$

$$\int_{t_1}^{t_2}\left[\int_0^L m(x)\dot{u}^t(x,t)\delta\dot{u}^t dx - \int_0^L EI(x)u''(x,t)\delta u'' dx + N\int_0^L u'(x,t)\delta u' dx\right] \tag{2.23}$$

$$\dot{u}^t = \dot{u} + \dot{u}_g \qquad u'' = \Psi''Z \qquad u' = \psi'Z$$
$$\dot{u} = \psi\dot{Z} \qquad \delta\dot{u}^t = \delta\dot{u} \qquad \delta u'' = \psi''\delta Z \tag{2.24}$$
$$\delta\dot{u} = \psi\delta\dot{Z}$$

Substituting Eq. 2.24 into 2.23

$$\int_{t_1}^{t_2}\left[\dot{Z}\delta\dot{Z}\int_0^L m(x)\psi^2 dx + \delta\dot{Z}\dot{u}_g(t)\int_0^L m(x)\psi dx - Z\delta Z\int_0^L EI(x)(\psi'')^2 dx + NZ\delta Z\int_0^L(\psi')^2 dx\right]dt = 0$$

integrating by parts

$$\int_{t_1}^{t_2}\left[\ddot{Z}\int_0^L m(x)\psi^2 dx + Z\int_0^L EI(x)(\psi'')^2 dx + ZN\int_0^L(\psi')^2 dx - \ddot{u}_g\int_0^L m(x)\psi dx\right]\delta Z dt = 0$$

$$\int_{t_1}^{t_2}\left[m * \ddot{Z} + k * Z - k_G Z - P * (t)\right]\delta Z dt = 0$$

since $\delta Z \neq 0$

$$\therefore m * \ddot{Z} + k * Z - k_G^* Z = P * (t) \tag{2.25}$$

where

$$m^* = \int_0^L m(x)\psi^2 dx = \text{generalized mass}$$

$$k^* = \int_0^L EI(x)(\psi'')^2 dx = \text{generalized stiffness}$$

$$k_G^* = N\int_0^L (\psi')^2 dx = \text{generalized geometric stiffness}$$

$$P^* = -\ddot{u}_g\int_0^L m(x)\psi dx = \text{generalized load}$$

Equation 2.25 can be written as

$$m * \ddot{Z}(t) + \bar{k} * Z(t) = P * (t) \quad , \bar{k}^* = k^* - k_G^* \tag{2.26}$$

to get critical buckling load put

$$\bar{k}^* = k^* - k_G^* = 0 \tag{2.27}$$

$$\therefore N_{cr} = \frac{\int_0^L EI(x)(\psi'')^2 dx}{\int_0^L (\psi')^2 dx} \tag{2.28}$$

Problem: assume $\psi(x) = 1 - \cos\frac{\pi x}{2L}$, find equation of motion and N_{cr}

2.5.3. Expression for Generalized System Properties

The Equation of motion for any SDOF system, no mater haw complex, can always be reduced to the form:

$$m^* \ddot{Z}(t) + c^* \dot{Z}(t) + \bar{k}^* Z(T) = P^*(T) \tag{2.29}$$

and the displacement

$$u(x,t) = \psi(x) Z(t)$$

$$\therefore m^* = \int_0^L m(x)[\psi(x)]^2 dx + \sum m_i \psi_i^2 + \sum I_i (\psi'_i)^2 \tag{2.29a}$$

where: $m(x) \equiv$ distributed mass & m_i & $I_i \equiv$ rigid body masses

$$c^* = \int_0^L c(x)[\psi(x)]^2 dx + \sum c_i \psi_i^2 \tag{2.29b}$$

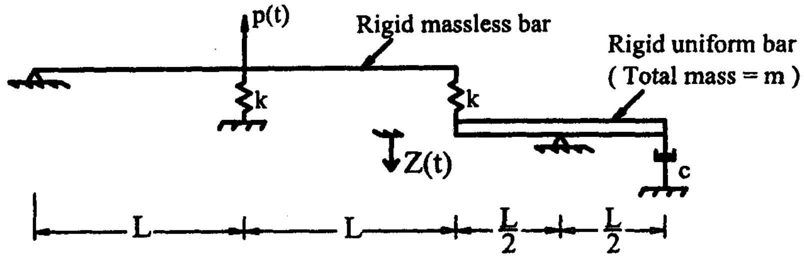
where $c(x) \equiv$ distributed damping and c_i = local dampers

$$k^* = \underbrace{\int_0^L k(x)[\psi(x)]^2 dx}_{\text{distributed spring}} + \underbrace{\int_0^L EI(x)[\psi''(x)]^2 dx}_{\text{flexural stiffness}} + \underbrace{\sum k_i \psi_i^2}_{\text{local springs}} \tag{2.29c}$$

$$k_G^* = \int_0^L N(x)[\psi'(x)]^2 dx$$

$$P^*(t) = \underbrace{\int_0^L P(x,t)\psi(x) dx}_{\text{distributed load}} + \underbrace{\sum P_i \psi_i}_{\text{concentrated loads}}$$

Problems



2.1 Determine **m*, c*, k* & P* (t)** with respect to the displacement coordinate **Z(t)**

Hint: The system has only one degree of freedom because the springs completely control the relative motion of the two rigid bars.

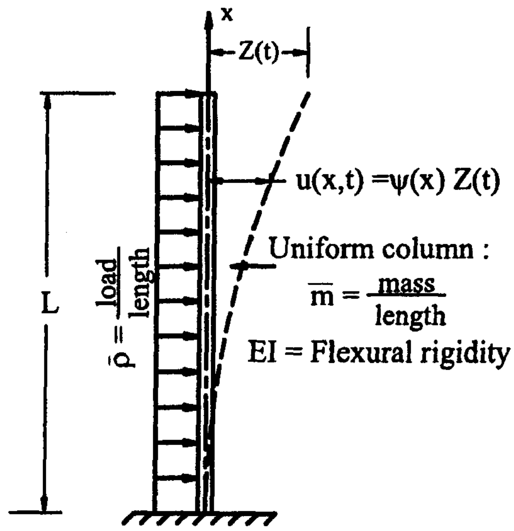
2.2 $\psi(x) = \left(\frac{x}{L}\right)^2 \left(\frac{3}{2} - \frac{x}{2L}\right)$

a) $N = 0$

b) $N = N$ applied at top

c) $N(x) = N\left(1 - \frac{x}{L}\right)$

find: **m*, c*, $\bar{k}$*, P***



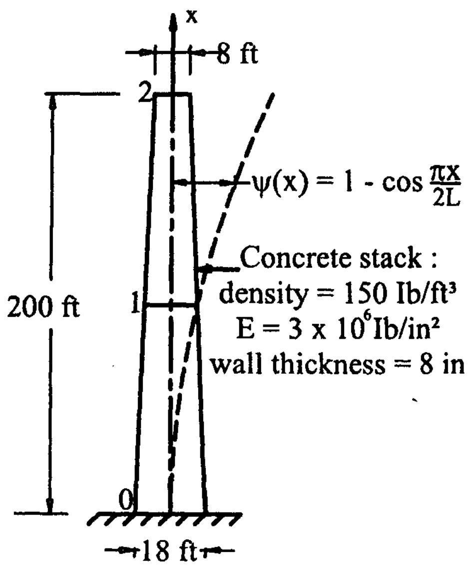
2.3 Compute **m*, k***

use Simpson's rule to evaluate the integrals including in the summations the integrand values for the bottom, middle and top sections.

For example

$$m^* = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$

where, $y_i = m_i\ \psi_i^2$ evaluated at level i



CHAPTER (3)

FREE VIBRATION RESPONSE OF SDOF

3.1 SOLUTION OF THE EQUATION OF MOTION

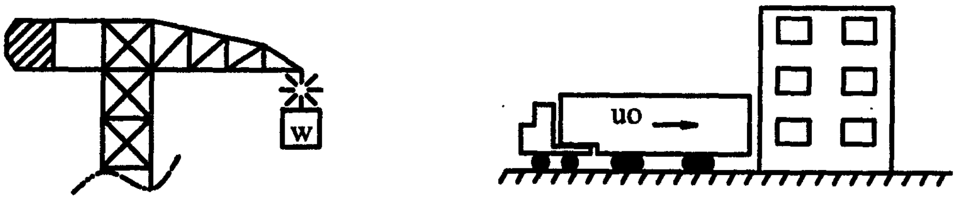
The Equation of motion of a simple spring mass system with damping, as shown Figure 2.1, is

$$m\ddot{y} + c\dot{y} + ky = P(t) \tag{3.1}$$

The solution of Equation 3.1 will be obtained by considering homogeneous equation with the right side set equal to zero.

$$m\ddot{y} + c\dot{y} + ky = 0 \tag{3.2}$$

Motion taking place with the applied force set equal to zero are called *Free Vibrations*. Various influences may cause free vibration response (or response to initial conditions), Figure 3.1 show what we mean by initial conditions that result in dynamic responses.



(a) Suddenly falling load (b) Impact load

Figure 3.1 Initial condition influences.

In the first case, a crane suddenly drops its load and rebounds from an initial condition of static-load displacement. In the second influence, a truck backs into a building at some initial velocity. Consequently, the building and the body of the truck respond dynamically.

3.2 UNDAMPED FREE VIBRATIONS

If the system is undamped, i.e., if $c = 0$, then Equation 3.2 becomes

$$m\ddot{y} + ky = 0 \tag{3.3}$$

in which $y = y(t)$ = displacement, $\ddot{y} = \frac{d^2y}{dt^2}$ = acceleration, t = time in seconds. Equation 3.3 is an ordinary differential Equation of the second order, homogeneous with constant coefficients. The solution of this Equation is of the form

$$y(t) = G\, e^{\lambda t} \tag{3.4}$$

Substituting Equation 3.4 into Equation 3.3 leads to

$$Gm\lambda^2 e^{\lambda t} + Gke^{\lambda t} = o \tag{3.5}$$

or $$\lambda^2 + \frac{k}{m} = o \tag{3.6}$$

Dividing by **m** and introducing the notation

$$\omega^2 = \frac{k}{m} \tag{3.7}$$

Equation 3.6 becomes

$$\lambda^2 = -\frac{k}{m} = -\omega^2 \tag{3.8}$$

i.e. $$\lambda = \pm i\omega \tag{3.9}$$

Thus the response given by Equation 3.4 is

$$y(t) = G_1 e^{i\omega t} + G_2 e^{-i\omega t} \tag{3.10}$$

Equation 3.10 can be put in a more convenient from by introducing Euler's Equation.

$$e^{\pm i\omega t} = \cos\omega t + i\sin\omega t \tag{3.11}$$

The result may be written as

$$y(t) = A\cos\omega t + B\sin\omega t \tag{3.12}$$

and $$\dot{y}(t) = -\omega A\sin\omega t + B\omega\cos\omega t \tag{3.13}$$

$$\left.\begin{aligned} \therefore y(o) &= A + o \\ \dot{y}(o) &= 0 + B\omega \end{aligned}\right\} \tag{3.14}$$

where, ω is the eircular frequency of the motion and constants **A** and **B** may be expressed in terms of the initial conditions, i.e. the displacement **y(o)** and velocity $\dot{y}(o)$ at time **t** = **0** which initiated the free vibration of the system.

From which $A = y(o)$, $B = \dfrac{\dot{y}(o)}{\omega}$

Thus Equation 3.12 becomes

$$\boxed{y(t) = y(o)\cos\omega t + \frac{\dot{y}(o)}{\omega}\sin\omega t} \tag{3.15}$$

The solution given by Equation 3.15 represents a simple harmonic motion (SHM) and is portrayed graphically in Figure 3.2

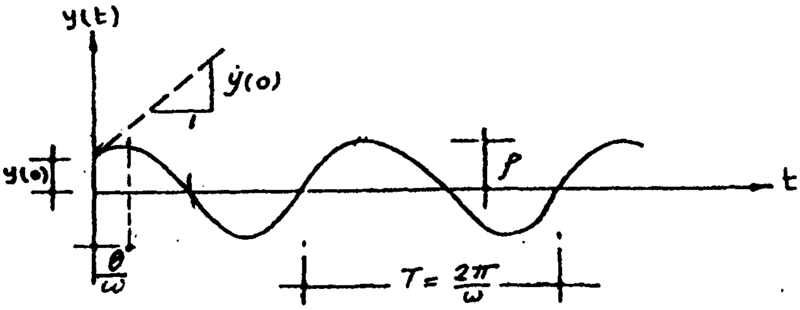


Figure 3.2a Undamped free vibration response.

Equation 3.15 can be rewritten as

$$y(t) = \rho\cos(\omega t - \theta)$$

or

$$\boxed{y(t) = \rho\cos\omega\left(t - \frac{\theta}{\omega}\right)} \tag{3.16}$$

where, the amplitude of motion is

$$\rho = \sqrt{[y(o)]^2 + \left[\frac{\dot{y}(o)}{\omega}\right]^2} \tag{3.17}$$

and the phase angle

$$\theta = \tan^{-1}\frac{\dot{y}(o)}{\omega y(o)} \tag{3.18}$$

Figure 3.2b shows that phase angle θ represents the angular distance by which the resultant motion lags behind the cosine term in the response.

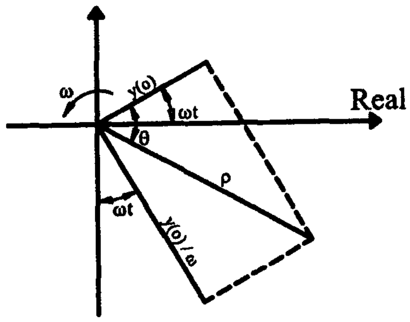


Figure 3.2b Rotating-Vector representation of free vibration.

Equation 3.7 is one of the basic expressions in dynamics as the natural frequency is a fundamental characteristic of a structure. The circular frequency is measured in radians per unit of time. The cyclic frequency, f, which is usually referred to merely as the frequency of the motion is given by

$$f = \frac{\omega}{2\pi} \qquad \text{HZ (cycle/sec)} \tag{3.19}$$

and its reciprocal is called the period T,

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \tag{3.20}$$

Equation 3.19 can be used to estimate the natural frequency from the static deflection ($y_{st.} = \frac{w}{k} = \frac{mg}{k}$) under the weight of the structure (w = mg) as follows

then $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\sqrt{\frac{g}{g}} = \frac{\sqrt{g}}{2\pi}\sqrt{\frac{k}{mg}} = \frac{\sqrt{g}}{2\pi}\frac{1}{\sqrt{y_{st.}}}$

$$\boxed{f \simeq \frac{5}{\sqrt{y_{st.}}}} \tag{3.21}$$

in which, $y_{st.}$ is y in cm.

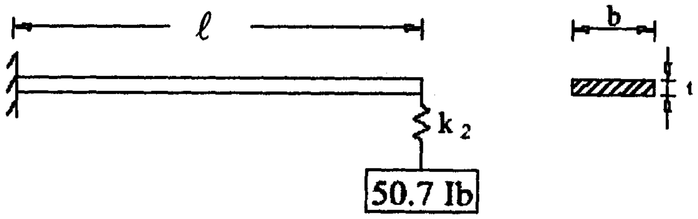
Empirical formulae for the natural frequency of multi story buildings are also available. One such formula is

$$\boxed{f = \frac{10}{N}} \tag{3.22}$$

where, N is the number of stories of average height.

Solved Examples

Example (1):



Determine the natural frequency of the system shown in the Figure consisting of a weight of 50.7 1b attached to a horizontal cantilever beam through the coil spring k_2. The beam has a thickness **t** = 1/4``, a width **b** = 1`` modulus of elasticity **E** = 30 $\times 10^6$ psi, and a length ℓ = 12.5``. The coil spring has a stiffness $\mathbf{k_2}$ = 10.69 1b/in.

Solution:

Stiffness of the cantilever $\mathbf{k_1} = \dfrac{\mathbf{3EI}}{\ell^3}$

$$\mathbf{k_1} = \frac{\mathbf{3 \times 30 \times 10^6 x1x\left(\frac{1}{4}\right)^3}}{\mathbf{(12.5)^3 12}} = \mathbf{60} \ \ 1b/in$$

since the cantilever and the coil spring are connected as springs in series

$$\therefore \frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$k_e = 9.07 \ \ lb/in$$

The natural frequency of the system

$$\omega = \sqrt{k_e / m} = \sqrt{k_e . g / w}$$

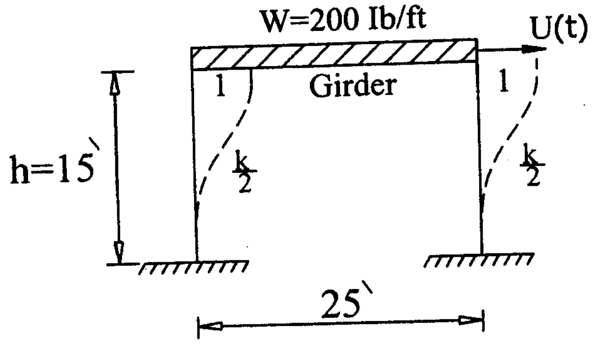
$$= \sqrt{9.07 \times 386 / 50.7}$$

$$= 8.31 \qquad rad/sec$$

$$f = \frac{\omega}{2\pi}$$

$$= 1.32 \qquad cycle/sec$$

Example (2):



Columns: $\mathbf{I} = 82.5\ in^4$
$\mathbf{E} = 30 \times 10^6\ lb/in^2$

The rectangular plane frame shown in the Figure has a very stiff mass **m** and rather flexible columns. The masses of the columns are negligible. Find the values of **f** and **T** for this structure treating it as a SDF system.

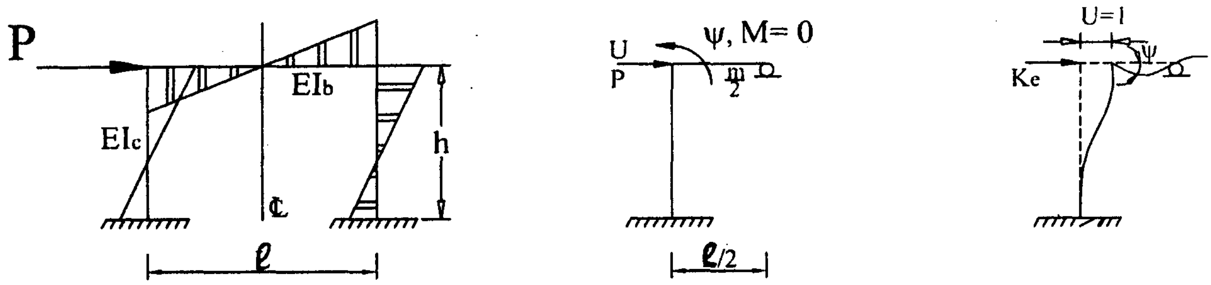
Solution:
By translating the girder a unit distance **u** = 1, the stiffness constant is

$$\mathbf{k = 2 \times \frac{12EI}{h^3} = \frac{2 \times 12 \times 30 \times 10^6 \times 82.5}{(15 \times 12)^3} = 10185\ lb/in}$$

$$\therefore \mathbf{f = \frac{1}{2\pi}\sqrt{\frac{kg}{w}} = \frac{1}{2\pi}\sqrt{\frac{10185 \times 386}{200 \times 25}} = 4.46\ cps}$$

$$\mathbf{T = \frac{1}{f} = 0.224\ sec.}$$

Such building is called ***shear building*** and a one story building has one DOF, a two story building has 2 DOF etc. If the girder is not rigid (stiff), each joint is subjected to horizontal translation u and rotation ψ and thus has two DOF. However, because mass moments of inertia of lumped masses can be neglected, the response may again be treated as SDOF, if the structure horizontal stiffness is evaluated allowing for joint rotation. Replace frame by its half because there is no bending moment on the centerline. Then, the stiffness constants associated with horizontal translation **u** and rotation ψ are



a) B.M.D *b) half the frame* *c) horizontal stiffness*

Figure 3.3 Modeling one story building (frame) as SDOF.

$$\mathbf{kuu = \frac{12EI}{h^3}} \quad , \quad \mathbf{k\psi\psi = \frac{4EI_c}{h} + \frac{3EI_b}{\lambda/2}}$$

$$\mathbf{ku\psi = k\psi u = \frac{6EI}{h^2}}$$

Equilibrium of the joint in the direction of **u** and ψ requires

$$\mathbf{\begin{Bmatrix} P \\ M \end{Bmatrix} = \begin{bmatrix} kuu & ku\psi \\ k\psi u & k\psi\psi \end{bmatrix} \begin{Bmatrix} u \\ \psi \end{Bmatrix}} \tag{3.23}$$

For stiffness of the whole building $\mathbf{k_e}$, put $\mathbf{P} = \mathbf{k_e} / 2$, $\mathbf{M} = o$, $\mathbf{u} = 1$. Substituting these values in Equation 3.23.

$$\mathbf{\therefore P = \frac{k_e}{2} = kuu(1) + ku\psi(\psi)} \tag{3.24}$$

and $\mathbf{0 = k\psi u(1) + k\psi\psi(\psi)}$

$$\mathbf{\therefore \psi = -k\psi u / k\psi\psi} \tag{3.25}$$

substituting Eq 3.25 into Equation 3.24

$$\mathbf{\therefore k_e = 2(kuu - k^2\psi u / k\psi\psi)}$$

the natural frequency is

$$\omega = \sqrt{\frac{\mathbf{k_e}}{\mathbf{m}}}$$

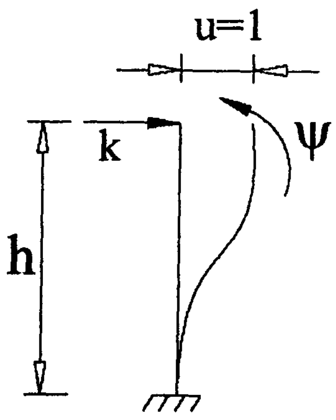
with more degrees of freedom, this process of eliminating rotations is carried out in matrix form and is called ***matrix condensation***.

Free standing column

$$E\,I_b = 0$$

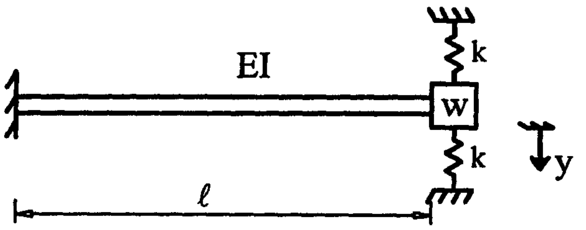
and condensed horizontal stiffness is

$$k = \frac{12EI}{h^3} - \frac{(6EI)^2}{h^4}\frac{h}{4EI}$$
$$= \frac{3EI}{h^3}$$



Problems

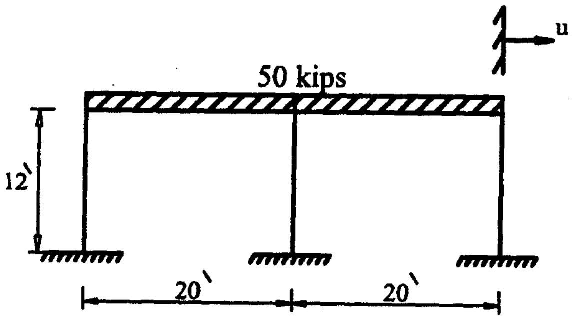
3.1.



Determine the natural period for the shown system. Assume that the beam and springs are massless.

$\ell = 100$ in , $EI = 10^8$ lbin^2 $w = 3000$ lb and $k = 2000$ lb / in

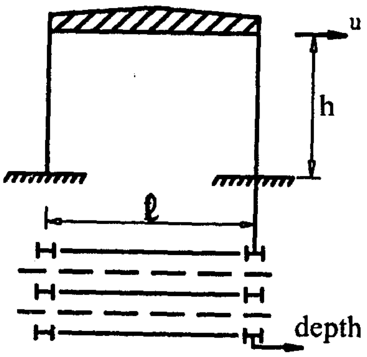
3.2



Columns : $I = 90$ in^4
$E = 30 \times 10$ Ib/in^2

Determine the natural frequency for horizontal motion of the steel frame shown. Assume the horizontal girder to be infinitely rigid and neglect the mass of columns.

3.3. Each section of the shown shear building is supported by two columns.



Each column has depth = 600 mm

$I = 560 \times 10^6$ mm^4
$E = 2 \times 10^5$ Mpa
$h = 5$ m

The participating mass of the structure is
$m = 30{,}000$ kg (for one bay)

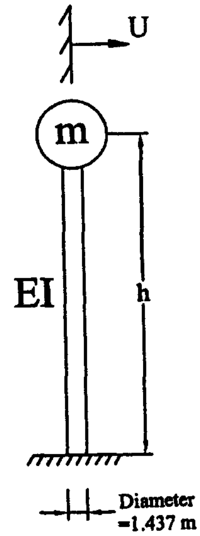
Calculate the natural frequencies assuming fixed-fixed columns and also hinges at the top of the columns.

3.4. Analyze the same building in problem 3.3 but allow for girder elasticity.

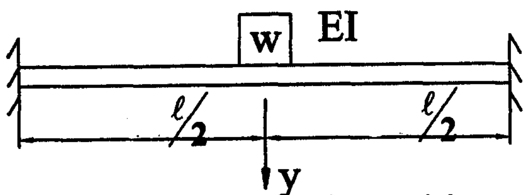
EI_{girder} EI_{column}, $\ell = h$

3.5. A vertical pole of height **h** and flexural rigidity **EI** carries a mass m at its top as shown. Find the Natural frequency of the system neglecting the weight Of the pole.

$m = 68{,}000$ kg, $E = 2 \times 10^5$ Mpa , $h = 152.4$ m and $I = 3.348$ m^4



3.6



If the initial displacement and the initial velocity of the weight are, respectively $y_o = 0.5$ in and $\dot{y}_o = 15$ in/sec, determine the displacement, velocity and acceleration of, W , when time $t = 2$ sec. $\ell = 120$ inch. $EI = 10^9$ lb. inch^2 $W = 5000$ 1b

Hint:

$$\text{Mpa} \equiv \text{MN}/\text{m}^2 = 1000 \ \text{kN/m}^2 = 10^6 \ \frac{\text{N}}{\text{m}^2}$$

when mass = slug = $\dfrac{\text{1b.sec}^2}{\text{in}}$ then $k \equiv \text{1b/in}$

when mass ≡ kg then $k \equiv \text{N}/\text{m}$, $\text{N} \equiv \dfrac{\text{kg.m}}{\text{sec}^2}$

$$\text{then } \omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{\text{kg.m}}{\text{sec}^2.\text{m}}\frac{1}{\text{kg}}} \equiv \sqrt{\frac{1}{\text{sec}^2}} \equiv \text{rad/sec}$$

if weight W is given in kg

then $k \equiv \dfrac{\text{kg}}{\text{cm}}$ and mass $= \dfrac{W}{981}$

3.3 DAMPED FREE VIBRATION

If damping is present, the equation of motion is (Equation 3.2)

$$m\ddot{y} + c\dot{y} + ky = o \tag{3.26}$$

Again this is an ordinary differential equation with constant coefficients and it is homogeneous. A particular solution is

$$y = Ge^{\lambda t} \tag{3.27}$$

substituting Equation 3.27 into Equation 3.26 yields

$$Gm\lambda^2 e^{\lambda t} + Gc\lambda e^{\lambda t} + Gke^{\lambda t} = o \tag{3.28}$$

or
$$m\lambda^2 + c\lambda + k = o \tag{3.29}$$

Dividing by **m** and using the notation $\omega^2 = \dfrac{k}{m}$

then
$$\lambda^2 + \frac{c}{m}\lambda + \omega^2 = o \tag{3.30}$$

This is an algebraic equation, the roots of which are

$$\lambda_{12} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega^2} \tag{3.31}$$

Three types of motion are presented by this expression, according to the quantity under the square-root sign is positive, negative or zero. That is to say: supercritical damping, subcritical damping and critical damping.

a. Critical Damping

If the radical in Equation 3.31 is set equal to zero, then $\dfrac{c}{2m} = \omega$, thus the critical damping ratio is

$$\boxed{c_{cr} = 2m\omega} \tag{3.32}$$

Also Equation 3.31 yields only one root

$$\lambda = \frac{-c}{2m} = -\omega \tag{3.33}$$

Therefore another particular solution must be added, such as $G_2\, te^{-\omega t}$. Then the general solution is

$$y = (G_1 + G_2 t)e^{-\omega t}$$

With the initial conditions $y(o)$ and $\dot{y}(o)$ then,

$$G_1 = y(o) \quad , \qquad G_2 = \dot{y}(o) + y(o).\omega$$

and the general solution becomes

$$y = e^{-\omega t}\left[y(o)(1+\omega t) + \dot{y}(o)t\right] \qquad (3.34)$$

This motion is not vibratory due to the exponential decay term of Equation 3.34.

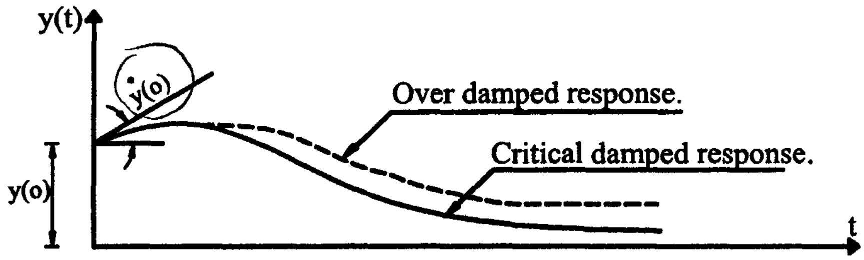


Figure 3.4 Free vibration response with critical and over damping.

One useful definition of the critically damped condition is that "it is the smallest amount of damping for which no oscillation occurs in the free response".

b. Under damped system

If damping is less than critical, then $c < 2m\omega$ thus the radical in Equation 3.31 is negative. Using the following definition.

$$D = \frac{c}{c_{cr}} = \frac{c}{2m\omega} \qquad (3.35)$$

In which, **D** is called the damping ratio, then Equation 3.31 can be rewritten as

$$\lambda = -\omega D \pm \sqrt{(\omega D)^2 - \omega^2}$$

or $\quad \lambda = -\omega D \pm i\omega\sqrt{1-D^2} \qquad (3.35`)$

or $\quad \lambda = -\omega D \pm i\omega`$

where, $\omega` = \sqrt{1-D^2}$ **= damped natural frequency,**

For typical structures **D** < 20%, then $\omega`$ differs very little from ω.

Real $(\lambda) = -\omega D$

$$\therefore D = \frac{-\text{Real}(\lambda)}{\omega}$$

Free vibration response of an under damped system can be evaluated by substituting Equation 3.35` into Equation 3.27, thus

$$y = Ge^{-(\omega D \pm i\omega\sqrt{1-D^2})\,t}$$

or $$y = e^{-\omega Dt}(G_1 e^{i\omega' t} + G_2 e^{-i\omega' t})$$

The first term represents a decaying function and the term in parentheses represents simple harmonic motion. Thus this expression can be rewritten as

$$y = e^{-\omega Dt}(A\cos\omega' t + B\sin\omega' t) \tag{3.36}$$

Finally, when the initial conditions **y (o)** and $\dot{y}(o)$ are introduced, then the constants **A & B** can be evaluated, giving

$$y = e^{-\omega Dt}\left[y(o)\cos\omega t + \frac{\dot{y}(o) + y(o)\omega D}{\omega'}\sin\omega' t\right] \tag{3.37}$$

or $$y = \rho e^{-\omega Dt}\cos(\omega t - \theta) \tag{3.38}$$

in which,

$$\rho = \sqrt{[y(o)]^2 + \left[\frac{\dot{y}(o) + y(o)\omega D}{\omega'}\right]^2} \tag{3.39}$$

$$\theta = \tan^{-1}\frac{\dot{y}(o) + y(o)\omega D}{\omega' y(o)} \tag{3.40}$$

Initial amplitude if the mass of the system is released from a stationary displaced position $[\dot{y}(o) = o. \;\&\; y(o) = \sqrt{}]$, then Equation 3.37 becomes

$$y = e^{-\omega Dt}\left[y(o)\cos\omega' t + \frac{y(o)\omega D}{\omega'}\sin\omega' t\right]$$

and if the damping is small, then the motion becomes

$$y \cong y(o)e^{-\omega Dt}\cos\omega' t \tag{3.41}$$

The time history of this motion is shown in Figure 3.5

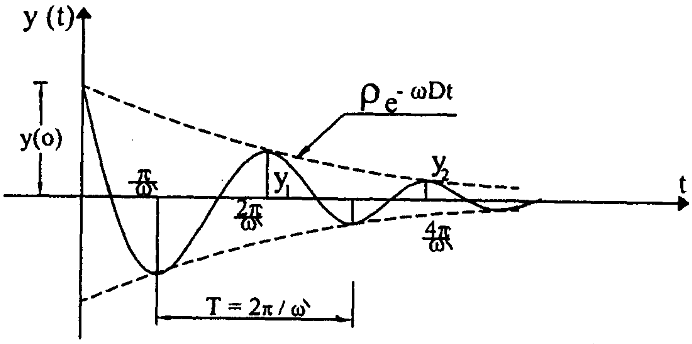


Figure 3.5 Motion triggered by initial amplitude.

Initial Velocity when only the initial velocity is applied, $\dot{y}(o) = \sqrt{}$, $y(o) = o$, then the resultant motion follows from Equation 3.37 as

$$y = \frac{\dot{y}(o)}{\omega'} e^{-\omega Dt} \sin \omega' t \tag{3.42}$$

This motion is a damped sine curve (Figure 3.6)

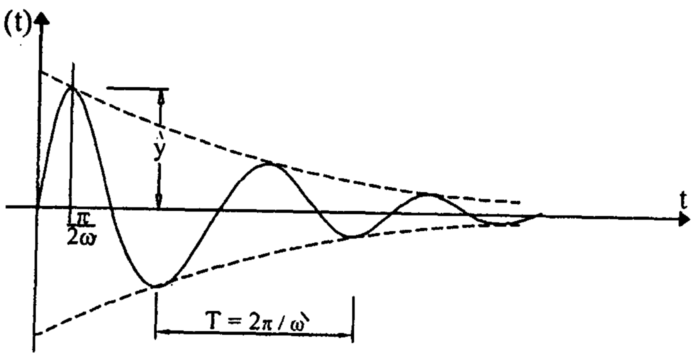


Figure 3.6 Motion triggered by initial velocity.

When the damping is small, the maximum amplitude is approximately

$$\hat{y} = \frac{\dot{y}(o)}{\omega'} \tag{3.43}$$

Initial velocity is important in collisions (Figure 3.7). If the duration of collision t_1 is much smaller than the natural period of the structure **T** (i.e $t_1 << T$), then the initial velocity of a structure whose mass is m, hit by moving mass m_o with velocity c_o is

$$\dot{y}(o) = c_0 (1 + Kr) \frac{mo}{m + mo} \tag{3.44}$$

in which, **Kr** = coefficient of collision (restitution).

For an elasto-plastic impact **Kr** = 0.5. With the initial velocity, $\dot{y}(o)$, estimated from Equation 3.44 and substituted in Equation 3.43, then the maximum response, $\hat{y}$, can be evaluated. Then the equivalent static force applied to the structure is

$$P_{eq} = k.\hat{y} \tag{3.45}$$

Problem

3.7. Consider that the mass at the top of the vertical pole of problem 3.5 is hit by a cruising airoplane of mass 1200 kg and flying at speed of 180 km/h. Calculate the maximum displacement at the top of the pole and the maximum stresses at its foot, assuming **Kr** = 0.5 and **D** = 0.01.

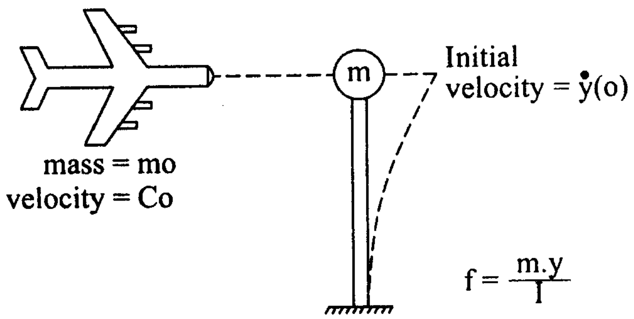


Figure 3.7 Collision of an airoplane with Mass 'm' at the top of a vertical pole.

3.4 LOGARITHMIC DECREMENT OF DAMPING

Consider any two successive peaks of viscously damped oscillation, that is y_n and y_{n+1}, where **n** = number of elapsed periods T (Figure 3.5). From Equation 3.27

$$y_n = y_1 = y_{(o)} e^{-\omega D \frac{2\pi}{\omega'}}$$

$$y_{n+1} = y_2 = y_{(o)} e^{-\omega D \frac{4\pi}{\omega'}}$$

then
$$\frac{y_n}{y_{n+1}} = e^{+2\pi D(\frac{\omega}{\omega'})} \tag{3.46}$$

Taking the natural logarithm (λn) of both sides of Equation 3.46 gives the logarithmic decrement δ

$$\delta = \lambda n \frac{y_n}{y_{n+1}} = 2\pi D \frac{\omega}{\omega'}$$

$$= \frac{2\pi D}{\sqrt{1 - D^2}} \tag{3.47}$$

For small damping

$$\delta = 2\pi D \tag{3.48}$$

The Logarithmic decrement is used to establish the damping of a system from the record of free vibration. Better accuracy can be obtained by considering peaks which are several cycles (periods) apart, say **M** cycles, then

$$\delta = \ell n \frac{y_n}{y_{n+M}} = 2\pi M \frac{\omega}{\omega^`} D \tag{3.49}$$

c. Over damped system

In this case D > 1 and Equation 3.31 yields the two real roots

$$\lambda_{12} = -\omega D \pm \omega\sqrt{D^2 - 1} = -\omega D \pm \omega^` \tag{3.50}$$

Substituting Equation 3.50 into Equation 3.27 leads to

$$y = e^{-\omega Dt}(G_1 e^{\omega t} + G_2 e^{-\omega^` t})$$

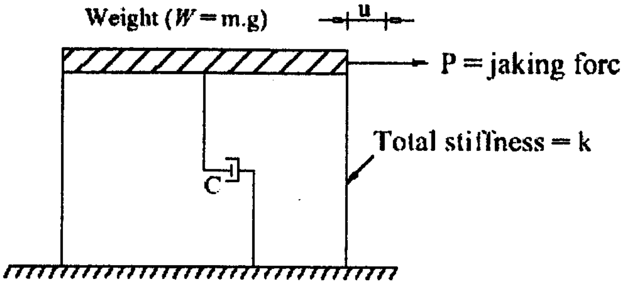
or $$y = e^{-\omega Dt}(A \sinh \omega^` t + B \cosh \omega^` t) \tag{3.51}$$

in which the constants **A** & **B** could be evaluated from initial conditions. The response of an over damped system (Equation 3.51) is not oscillatory, it is shown in Figure 3.4.

Example (3):

In order to evaluate the dynamic properties of the shown one story shear building, a free vibration test is made. It is found that a force of 20 kips is required to displace the girder 0.2 inch. After this initial displacement, it is observed that the maximum displacement in the return swing is only 0.16 inch and the period of this displacement cycle is **T** = 1.40 sec. Calculate the stiffness of the system **k**, natural frequency ω, weight of the girder **W**, damping ratio **D**. amplitude after 6 cycles and damping coefficient **c**.

Solution:



$$k = \frac{\text{force}}{\text{displacement}} = \frac{20}{0.2} = 100 \text{ kips/in}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{1.4} = 4.486 \text{ rad/sec.}$$

$$W = m.g = \frac{k}{\omega^2}.g$$

$$= \frac{100}{(4.486)^2} \times 386 = 1920 \text{ kips}$$

$$-\delta = \ell n \frac{.2}{.16} \cong 2\pi D = 0.223$$

$$\therefore D = \frac{\delta}{2\pi} = 3.55\%$$

- Amplitude after 6 cycles

$$\ell n \frac{.2}{y_6} = 2\pi(6)(D)$$

$$\therefore y_6 = \surd$$

- Damping coefficient

$$c = D\, c_{cr}$$

$$= D\, 2\, m\, \omega$$

$$= \frac{3.55}{100} \times 2 \times \frac{1920}{386} \times 4.486$$

$$= 1.584 \quad \text{kips.sec / in}$$

Problems

3.8. The weight W of the building of Example (3) is 200 kips and the building is set into free vibration by releasing it (at time $t = o$) from a displacement of 1.2 inch. If the maximum displacement on the return swing is 0.86 inch at time $\mathbf{t}$ = 0.64 sec., determine:

a. The lateral stiffness **k**.
b. The damping ratio **D**.
c. The damping coefficient **c**.

3.9. Assume that the mass and stiffness of the structure of Example (3) are as follows: **m** = 2 kip. s^2 / in, **k** = 40 kip /in. If the system is set into free vibration with the initial conditions **y (o)** = 0.7 inch and $\dot{\mathbf{y}}(\mathbf{o}) = \mathbf{5.6}$ **in / s**, determine the displacement at **t** = 1.0 sec, assuming.

a. **c** = 0 (undamped system).
b. **c** = 2.8 kip. s / in

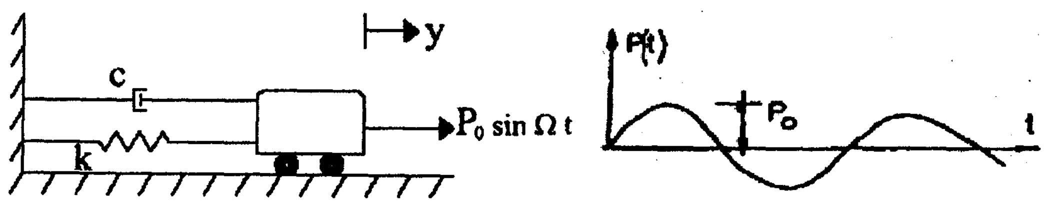
3.10. Repeat problem 3.6 assuming that the system has 15% of critical damping.

CHAPTER (4)

RESPONSE OF SDOF TO HARMONIC LOADING

Assume that the SDOF system shown in the figure is subjected to a harmonically varying load **P(t)** of amplitude P_o and circular frequency Ω. In this case the differential Equation of motion becomes:

$$m\ddot{y} + c\dot{y} + ky = P_o \sin \Omega t \tag{4.1}$$



4.1 UNDAMPED SYSTEM

In the absence of damping, the equation of motion becomes:

$$m\ddot{y} + ky = P_o \sin \Omega t \tag{4.2}$$

The solution of Equation 4.2 can be expressed as

$$y(t) = y_c(t) + y_p(t) \tag{4.3}$$

where, $y_c(t)$ is the *complementary solution* satisfying the homogeneous equation, i.e., Equation 4.2 with the right-hand side equal to zero; $y_p(t)$ is the *particular* solution based on the solution satisfying the non homogeneous differential Equation 4.2.
the complementary solution is

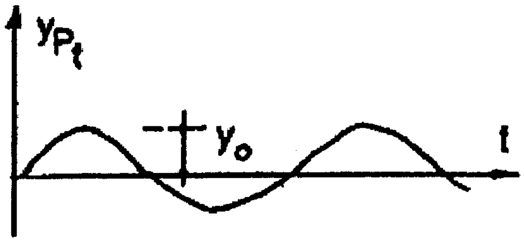
$$y_c(t) = A \sin \omega t + B \cos \omega t \tag{4.4}$$

where, $\omega = \sqrt{\dfrac{k}{m}}$

Then, the particular solution can be obtained by assuming that the response to the harmonic loading is, also, harmonic and in phase with the loading, thus

$$y_p(t) = y_o \sin \Omega t \tag{4.5}$$

where, y_o is the peak value of the particular solution.



Particular Solution for undamped system

Substituting Equation 4.5 into Equation 4.2 leads to

$$-m\Omega^2 y_0 \sin\Omega t + k y_0 \sin\Omega t = P_0 \sin\Omega t$$

or $$(k - m\Omega^2) y_0 = P_0 \qquad (4.6)$$

Dividing by **k** and noting that $\frac{k}{m} = \omega^2$ results in

$$\left(1 - \frac{\Omega^2}{\omega^2}\right) y_0 = \frac{P_0}{k}$$

or $$y_0 = \frac{P_0}{k}\left(\frac{1}{1 - \frac{\Omega^2}{\omega^2}}\right)$$

or $$y_0 = \frac{P_0}{k}\left(\frac{1}{1 - \beta^2}\right) \qquad (4.7)$$

in which,

$\frac{P_o}{k} = y_{st}$ is static displacement which would be produced by the load **Po** applied statically and

$\frac{1}{1-\beta^2}$ is the Dynamic magnification factor (MF)

$$MF = 1 \text{ if } \quad \Omega = o$$
$$MF = \infty \text{ if } \quad \Omega = \omega$$

A system acted upon by external excitation of frequency coinciding whit the natural frequency is said to at *resonance.*

General Solution

The general solution to the harmonic excitation of the undamped system is then given by the combination of Equations 4.4, 4.5 and 4.7 with Equation 4.3; thus

$$y(t) = A\sin\omega t + B\cos\omega t + \frac{P_o}{k}\frac{1}{1-\frac{\Omega^2}{\omega^2}}\sin\Omega t \tag{4.8}$$

If the initial condition at time $t = o$ are taken as zero ($y(o) = o$ and $\dot{y}(o) = 0$), the constants of integration are

$$A = \frac{-P_o}{k}\frac{\frac{\Omega}{\omega}}{1-\frac{\Omega^2}{\omega^2}} \quad , \qquad B = o$$

Then response given by Equation 4.8 becomes

$$y(t) = \frac{P_o}{k}\frac{1}{1-\frac{\Omega^2}{\omega^2}}\left(\sin\Omega t - \frac{\Omega}{\omega}\sin\omega t\right) \tag{4.9}$$

Thus, the response is given by the superposition of two harmonic terms of different frequencies:

$\sin\Omega t$ = response component at frequency Ω of the applied load
≡ Steady state response

and $\frac{\Omega}{\omega}\sin\omega t$ = response component at natural frequency
≡ free vibration effect induced by the initial conditions
≡ transient response because in a practical case, damping will cause this term to vanish.

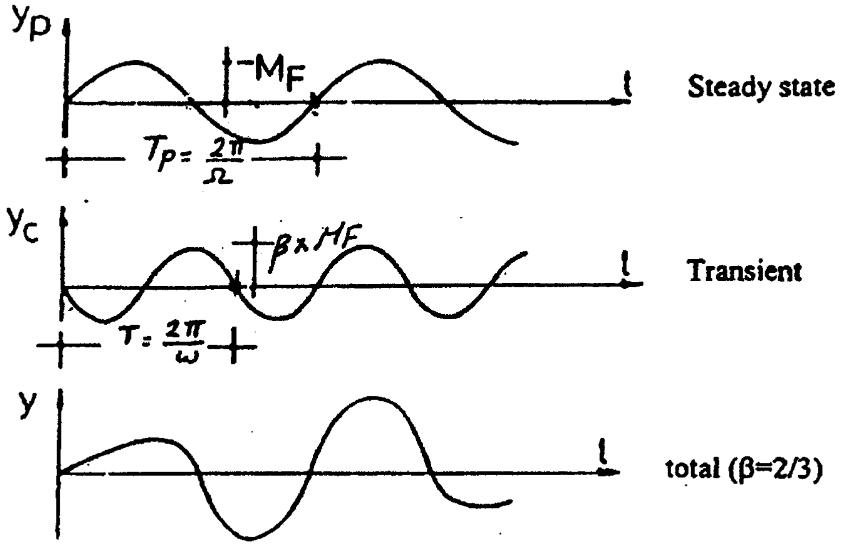


Figure 4.1 **Response to harmonic load from at-rest initial conditions.**

4.2 DAMPED SYSTEM

Dividing the equation of motion, Equation 4.1, by **m** and noting that $\frac{c}{m} = 2\omega D$ leads to

$$\ddot{y} + 2\omega D\dot{y} + \omega^2 y = \frac{P_o}{m}\sin\Omega t \tag{4.10}$$

The complete solution of this equation again consists of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$. The complementary solution of the equation is the damped free-vibration response given by Equation 3.36 (for $c < c_{cr}$) as

$$y_c(t) = e^{-\omega Dt}(A\cos\omega' t + B\sin\omega' t) \tag{4.11}$$

The particular solution to this harmonic loading is of the form

$$y_p(t) = y_1 \sin\Omega t + y_2 \cos\Omega t \tag{4.12}$$

In which the second term is required because, in general, the response of a damped system is not in phase with the loading.

Substituting Equation 4.12 into Equation 4.10 leads to

$$(-y_1\Omega^2 - 2\omega D y_2\Omega + y_1\omega^2)\sin\Omega t + (-y_2\Omega^2 + 2\omega D y_1\Omega + y_2\omega^2)\cos\Omega t = \frac{P_o}{m}\sin\Omega t$$

Separating the multiples of **sin Ω t** from the multiples of **cos Ωt** leads to

$$(-y_1\Omega^2 - 2\omega D y_2\Omega + y_1\omega^2)\sin\Omega t = \frac{P_o}{m}\sin\Omega t$$

$$\text{and} \quad (-y_2\Omega^2 + 2\omega D y_1\Omega + y_2\omega^2) = 0$$

$$\left.\begin{aligned} &\text{or} \quad y_1\left(1-\frac{\Omega^2}{\omega^2}\right) - y_2\left(2D\frac{\Omega}{\omega}\right) = \frac{P_o}{k} \\ &\text{and} \quad y_2\left(1-\frac{\Omega^2}{\omega^2}\right) + y_1\left(2D\frac{\Omega}{\omega}\right) = o \end{aligned}\right\} \tag{4.13}$$

Solving these equations simultaneously results in

$$\left.\begin{aligned} y_1 &= \frac{P_o}{k}\frac{1-\dfrac{\Omega^2}{\omega^2}}{\left(1-\dfrac{\Omega^2}{\omega^2}\right)^2+\left(2D\dfrac{\Omega}{\omega}\right)^2} \\ y_2 &= \frac{P_o}{k}\frac{-2D\dfrac{\Omega}{\omega}}{\left(1-\dfrac{\Omega^2}{\omega^2}\right)^2+\left(2D\dfrac{\Omega}{\omega}\right)^2} \end{aligned}\right\} \tag{4.14}$$

Substituting Equation 4.14 into Equation 4.12 and combining $y_c(t)$ with $y_p(t)$ yields the general solution

$$\begin{aligned} y(t) &= e^{-\omega Dt}(A\cos\omega' t + B\sin\omega' t) \\ &+ \frac{P_o}{k}\frac{1}{\left(1-\dfrac{\Omega^2}{\omega^2}\right)^2+\left(2D\dfrac{\Omega}{\omega}\right)^2}\left[\left(1-\frac{\Omega^2}{\omega^2}\right)\sin\Omega t - 2D\frac{\Omega}{\omega}\cos\Omega t\right] \end{aligned} \tag{4.15}$$

The 1st term represents the ***transient response*** which damps out quickly because of the presence of the exponential term $e^{-\omega Dt}$ therefore, the response is given by only the 2nd term which is the ***steady state response.*** Then

$y(t) \simeq$ steady state response at the frequency of the applied loading, Ω, but out of phase with it.

or $$y(t) = \rho\sin(\Omega t - \theta) \tag{4.16}$$

in which, ρ is the amplitude of the steady-state response.

$$\rho = \frac{P_o}{k}\frac{1}{\sqrt{\left(1-\dfrac{\Omega^2}{\omega^2}\right)^2+\left(2D\dfrac{\Omega}{\omega}\right)^2}} \tag{4.17a}$$

and the phase angle, θ, by which the response lags behind the applied load is

$$\theta = \tan^{-1}\frac{2D\dfrac{\Omega}{\omega}}{1-\dfrac{\Omega^2}{\omega^2}} \tag{4.17b}$$

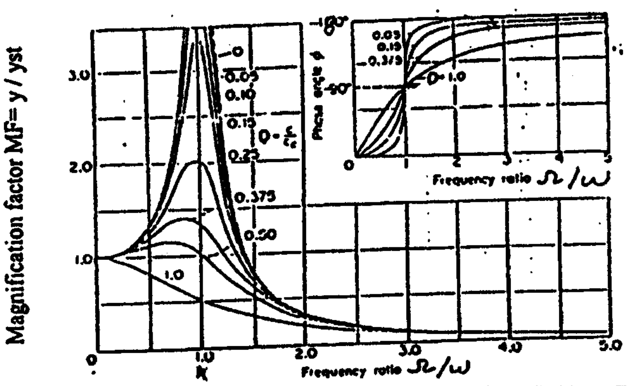
The dynamic magnification factor, MF, is the ratio of response to the static displacement produced by the load P_o:

$$MF = \frac{\rho}{P_o / k} = \frac{1}{\sqrt{\left(1-\frac{\Omega^2}{\omega^2}\right)^2 + \left(2D\frac{\Omega}{\omega}\right)^2}} \tag{4.18a}$$

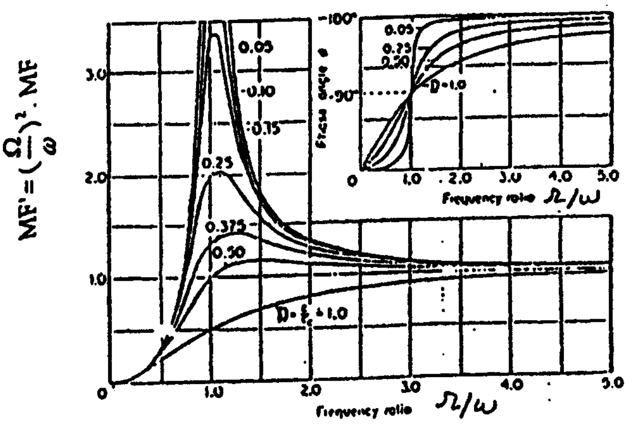
The variation of the dynamic magnification factor, **MF**, and the phase angle, **θ**, with frequency ratio $\frac{\Omega}{\omega}$ are shown in Figure 4.2. The factor **MF** starts from 1.0, then at resonance ($\Omega=\omega$) it becomes **MF=1/2D** and then diminishes and approaches zero as ω approaches infinity, With small damping, resonance response far exceeds the static response. The phase shift ranges from 0 to -180° and at resonance is -90° $(-\frac{\pi}{2})$ for all values of damping.

Examination of the response peaks reveals that the actual maximum of the magnification factor exceeds the value of **1/2D** and occurring at frequency

This maximum is
$$\left.\begin{aligned} \beta_{peak} &= \left(\frac{\Omega}{\omega}\right)_{peak} = \sqrt{1-2D^2} \\ MF_{max} &= \frac{1}{2D\sqrt{1-D^2}} \end{aligned}\right\} \text{for } D < \frac{1}{\sqrt{2}} \tag{4.18b}$$



a) Constant Force Excitation $P(t) = P_o \sin \Omega t$



b) Quadratic Excitation $P(t) = P'_o \Omega^2 \sin \Omega t$

Figure 4.2 **Dimensionless response to harmonic loads. (Dynamic amplification factor and phase shift)**

Quadratic Excitation

The above formulae and Figure 4.2a were derived for an excitation force whose amplitude P_o is constant. In many practical cases, the force amplitude is not constant but depends on the square of frequency. This is so with excitation stemming from centrifugal forces of unbalanced rotating masses, unbalance of reciprocating machines and harmonic ground motion. In this case

$$P(t) = P_o` \, \Omega^2 \sin \Omega t \tag{4.19}$$

Substitution of the amplitude $P`_o\Omega^2$ instead of P_o in Equation 4.17a gives

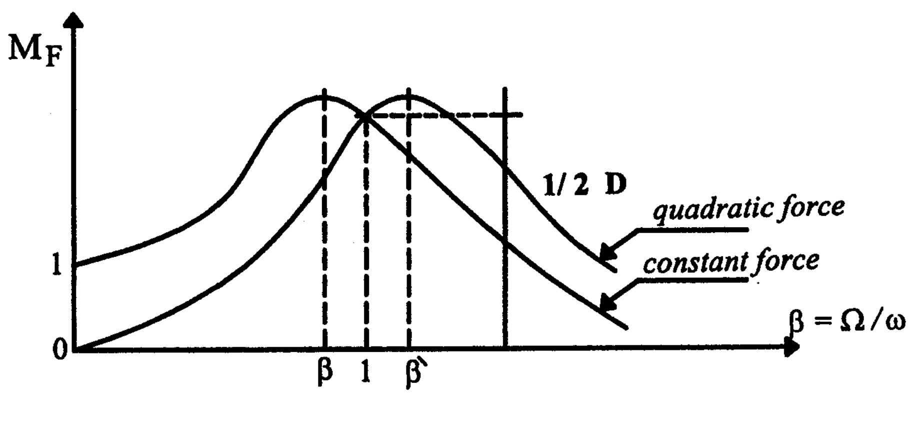
$$\begin{aligned} \rho &= \frac{P_o`\Omega^2}{k} MF \\ &= \frac{P_o`\Omega^2}{m.\omega^2} MF \\ &= \rho' MF' \end{aligned}$$

in which **MF`** is the dynamic magnification factor of quadratic excitation.

$$MF` = \left(\frac{\Omega}{\omega}\right)^2 MF \tag{4.20}$$

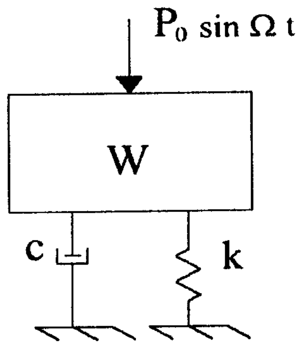
The variation in the dynamic magnification factor **MF'** with frequency is shown in Figure 4.2b. It starts from zero, grows to **1/2D** at resonance and finally drops to 1 at high frequency. The peak response occurs at frequency ratio.

$$\beta`peak = \frac{1}{\sqrt{1 - 2D^2}}$$



Example:

A machine of weight **W** = 53 kg is supported by a spring of stiffness **k** = 13 kg /cm and is restrained by a damper having a damping coefficient **c**=0.08 kg.s/cm. Find the maximum response of this machine due to harmonic load of amplitude $\mathbf{P_o}$ = 3.2 kg.



$$k=13 \ \text{kg/cm}=1300 \ \text{kg/m}=\frac{1300}{9.81} \ \text{N/m}$$

$$W=53 \ \text{kg}=\frac{531}{9.81} \ \text{N}$$

$$m=\frac{W}{9.81}=\frac{53}{9.81\times 9.81} \ \text{kgmass}$$

then

$$\omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{1300}{9.81}\times\frac{9.81\times 9.81}{53}}=1551 \ \text{rad/sec}$$

$$c_{cr}=2m\omega=2\times\frac{53}{981}\times 1551=1.676 \ \text{kg.s/cm}$$

$$D=\frac{c}{c_{cr}}=\frac{0.08}{1.676}=0.0477$$

$$y_{st}=\frac{P_o}{k}=\frac{2.2}{13}=0.169 \ \text{cm} \qquad , \qquad MF_{max}=\frac{1}{2D\sqrt{1-D^2}}=10.49$$

$$y_{max}=y_{st}.MF_{max}$$
$$=1.773 \ \text{cm}$$

hint:

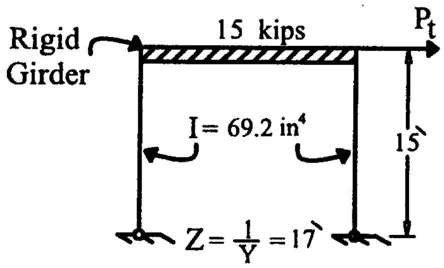
use $k \equiv N/m$ & $m \equiv kg$

or $k \equiv kg/cm$ & $m \equiv \frac{W}{g=981}$

Example:

The steel frame shown supports a machine which exerts a horizontal harmonic force **P(t)** at the girder's level. Assuming 5% of critical damping and **P(t)** = 200 sin 5.3 t. 1b determine:

a. The steady state amplitude of vibration.
b. Maximum shear force in the columns.
c. Maximum dynamic stress in the column.



$$k = 2 \times \frac{3EI}{\ell^3} = 2 \times \frac{3 \times 30 \times 10^6 \times 69.2}{(15 \times 12)^3} = 2136 \quad 1b/in$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2136}{15{,}000/386}} = 7.41 \quad rad/sec$$

$$\beta = \frac{\Omega}{\omega} = \frac{5.3}{7.41} = 0.715 \qquad , \quad y_{st} = \frac{P_o}{k} = \frac{200}{2136} in$$

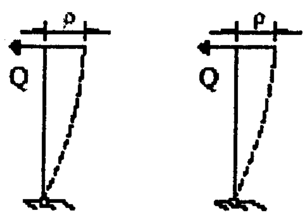
a. *Steady –state amplitude is*

$$\rho = \frac{P_o}{k} \frac{1}{\sqrt{(1-\beta)^2 + (2D\beta)^2}}$$

$$= \frac{200}{2136} \frac{1}{\sqrt{(1-.715^2)^2 + (2 \times .05 \times .715)^2}}$$

$$= 0.189 \quad in$$

b. *maximum shear in one column*



$$Q = \rho . \frac{3EI}{\ell^3}$$

$$= 0.189 \frac{3 \times 30 \times 10^6 \times 69.2}{(15 \times 12)^3} = 201.8 Ib$$

c. *maximum bending in the column*

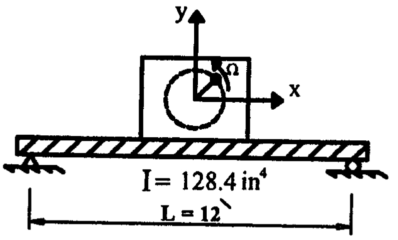
$$M_{max} = Q_{max} . \ell = 201.8 \times 15 \times 12$$
$$= 36{,}324 \quad 1b. in$$

maximum stress

$$\sigma_{max} = \frac{M_{max}}{Z} = \frac{36324}{17} = 2136 \ psi$$

Example:

A simple beam supports a machine having a mass **m** = 41.451 slug. The motor runs at 300 rpm and its rotor weight = 40 1b and it is out of balance by eccentricity **e** = 10 in. Calculate the amplitude of the steady slate response if the damping is assumed to be 10% of critical (neglect the mass of the beam).



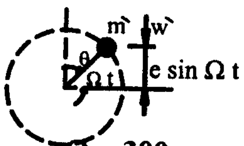
$$k = \frac{48EI}{\ell^3} = \frac{48 \times 30 \times 10^6 \times 128.4}{(12 \times 12)^3}$$

$$= 16.920 \quad 1b/in$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{61{,}970}{41.451}} = 38.65 \quad rad/s$$

$$\beta = \frac{\Omega}{\omega} = 0.813$$

$$P_o = m`.e.\Omega^2 = \frac{40}{386} \times 10 \times (31.41)^2 = 1022 \quad 1b$$



$$\Omega = 300 \quad rpm$$

$$= \frac{300}{60} \times 2\pi$$

$$\rho = \frac{P_o}{k}.MF \quad , \quad MF = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\beta D)^2}} = 2.66$$

$$= \frac{1022}{61.920}.MF = .044 \quad in$$

4.3 HARMONIC GROUND EXCITATION

Assume ground motion $u_g(t) = u_{go}\cos\Omega t$ then

$$\ddot{u}_g = -u_{go}\Omega^2\cos\Omega t,$$

The equation of motion of simplified model of Figure 2.5, in terms of relative displacement, **u**, is Equation 2.9

$$\begin{aligned} m\ddot{u} + c\dot{u} + ku &= -m\ddot{u}g \\ &= +mu_{go}\Omega^2\cos\Omega t \\ &= +P(t) \end{aligned}$$

in which. **P(t)** is the effective excitation force and its amplitude is

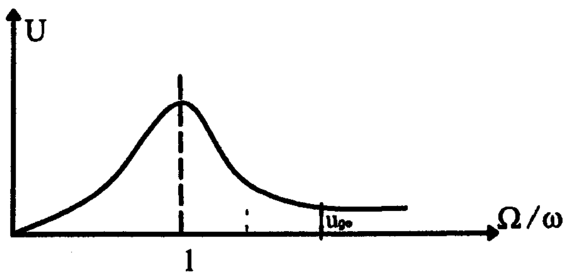
$$P_o = mu_{go}\Omega^2$$

Thus, harmonic ground motion represents a case of quadratic excitation. Then

$$\begin{aligned} \rho &= \frac{P_o}{k}MF \\ &= \frac{mu_{go}}{m\omega^2}\Omega^2 MF \\ &= u_{go}\left(\frac{\Omega}{\omega}\right)^2 MF \\ &= u_{go}MF' \end{aligned}$$

$$\text{and} \quad MF' = \left(\frac{\Omega}{\omega}\right)^2 MF$$

$$\begin{aligned} u^t &= u_g(t) + u(t) \\ &= u_{go}\cos\Omega t + \rho\cos(\Omega t + \theta) \end{aligned}$$



Problems

4.1 Predict the horizontal response of the one story building given in problem 3.3 to harmonic force **P(t)** resulting from a large static fan. Evaluate the stress in the foot the column.

$$P(t) = 0.002\Omega^2\sin\Omega, \quad \text{Damping ratio } D = 1\%$$

Calculate the response at $\Omega = 0.5\omega, \Omega = \omega, \Omega = 1.5\omega$. Consider fixed head column.

4.2 Predict the response of the building given in problem 3.3 to harmonic ground motion. Assume upper hinges, **D** =1%, u_{go}=0.025 cm. Calculate:

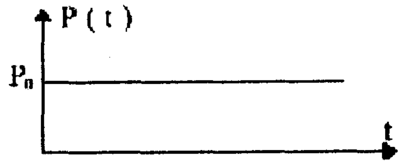
a. Motion of the roof both absolute and relative.

b. Stresses in the columns.

Analyze the vibration at $\Omega = \frac{\omega}{2}, \Omega = \omega \text{ and } \Omega = 2\omega$.

4.4 SUDDENY APPLIED STATIC LOAD

Equation of motion for undamped system



$$m\ddot{y} + ky = P_o$$

the particular solution is

$$y_p = \frac{P_o}{k} = \text{static deflection}$$

total solution is

$$y_t = y_c + y_p$$

$$= A\sin\omega t + B\cos\omega t + \frac{P_o}{k} \qquad (4.21)$$

constants A and B can be estimated from initial conditions, i.e., $y(o) = o$ and $\dot{y}(o) = o$ then

$$y(o) = o = o + B + \frac{P_o}{k}$$

$$\therefore B = -\frac{P_o}{k} \qquad (4.22)$$

$$\dot{y}(t) = A\omega\cos\omega t - B\omega\sin\omega t$$

$$\dot{y}(o) = o = A\omega \rightarrow A = o \qquad (4.23)$$

substituting Equations 4.22 and 4.23 into Equation 4.21

$$\therefore y_t = \frac{P_o}{k}(1 - \cos\omega t)$$

$$= y_{st}(1 - \cos\omega t) \qquad 4.24)$$

The maximum response can be determined by differentiating Equation 4.24 with respect to time and equating to zero: thus

$$\frac{dy}{dt} = o = \frac{P_o}{k}(\omega\sin\omega t)$$

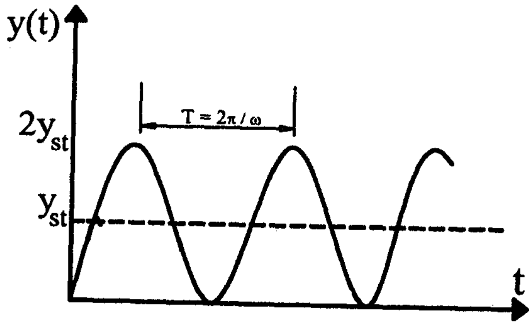
$$\therefore \sin\omega t = o$$

and hence $\omega t = o, \pi, 2\pi, \ldots n\pi$

or $t = o, \frac{\pi}{\omega}, \frac{2\pi}{\omega}, \ldots \frac{n\pi}{\omega}$

Then the maximum response is $y_{max} = 2y_{st.}$, i.e. twice as large as the static deflection under the load P_o.

The time history of the response given by Equation 4.24 is shown in Figure 4.3 for both damped and undamped system.



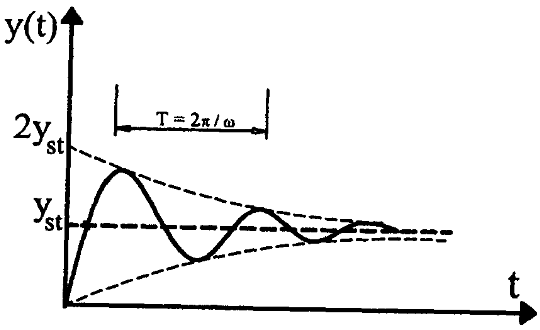


Figure 4.3 Response to suddenly applied static load.

Problems:

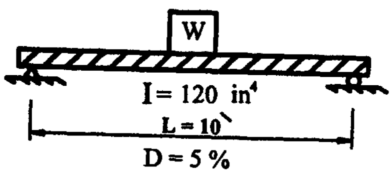
4.3. Prove that damped response to suddenly applied static load is

$$y(t) = \frac{P_o}{k}\left[1 - e^{-\omega Dt}(\cos.\omega' t + \frac{D}{\sqrt{1-D^2}}\sin\omega' t)\right]$$

and

$$y_{max} \cong \frac{P_o}{k}(1 + e^{-D\pi})$$

4.4. If a load **W** = 3680 1b is suddenly applied to a simply supported steel beam. Determine the maximum amplitude of the motion both damped and undamped.



4.5 VIBRATION ISOLATION

The idea of vibration isolation is to decrease the motion and the force transmitted to the structure, or the people. Two different classes of problems may be identified in which vibration isolation may be necessary:

(1) operating equipment or machine may generate oscillatory forces which could produce harmful vibration in the supporting structure (Figure 4.4a) or

(2) sensitive instruments may be supported by a structure which is vibrating appreciably (Figure 4.4b)

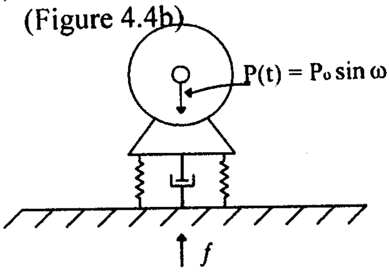


Figure 4.4a
SDOF vibration-isolation system
(applied loading).

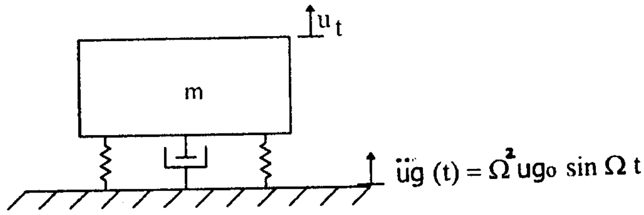


Figure 4.b
SDOF vibration-isolation system
(support excitation).

In the first situation, a rotating machine produces an oscillatory vertical force $P_o \sin \Omega t$ due to unbalance in its rotating parts. If the machine is mounted on a SDOF spring-damper support system, as shown, its steady state displacement response is

$$y(t) = \frac{P_o}{k} MF \sin(\Omega t - \theta) \tag{4.25}$$

thus, the force exerted against the base by the spring support, f_s, is

$$f_s = ky(t) = P_o MF \sin(\Omega t - \theta)$$

at the same time, the damping force on the base, f_D, is

$$\begin{aligned} f_D = c\dot{y} &= \frac{cP_o MF\Omega}{k}\cos(\Omega t - \theta) \\ &= \frac{2Dm\omega P_o MF\Omega}{m\omega^2}\cos(\Omega t - \theta) \\ &= 2DP_o \frac{\Omega}{\omega} MF \cos(\Omega t - \theta) \end{aligned}$$

since this force is 90° out of phase with the spring, then the amplitude of the base force, **f**, is

$$\begin{aligned} f_{max} &= \sqrt{f_{s,max}^2 + f_{D,max}^2} \\ &= P_o MF \sqrt{1 + (2D\beta)^2} \end{aligned} \tag{4.26}$$

Then, the ratio of the maximum base force to the applied-force amplitude, which is known as the transmissibility, **TR**, of the support system is

$$\mathbf{TR} = \frac{\mathbf{f_{max}}}{\mathbf{P_o}} = \mathbf{MF}\sqrt{\mathbf{1 + (2D\beta)^2}} \tag{4.27}$$

A plot of the transmissibility as a function of the frequency ratio, β, and damping ratio, **D**, is shown in Figure 4.5, all curves pass through the same point at frequency ratio of $\beta = \sqrt{2}$. Because of this characteristic, it is evident that the damping tends to reduce the effectiveness of a vibration-isolation system for frequencies greater than this critical ratio. The figure indicates that an isolation system is effective only for frequency ratio $\beta = \frac{\Omega}{\omega} > \sqrt{2}$ and damping is undesirable in this range.

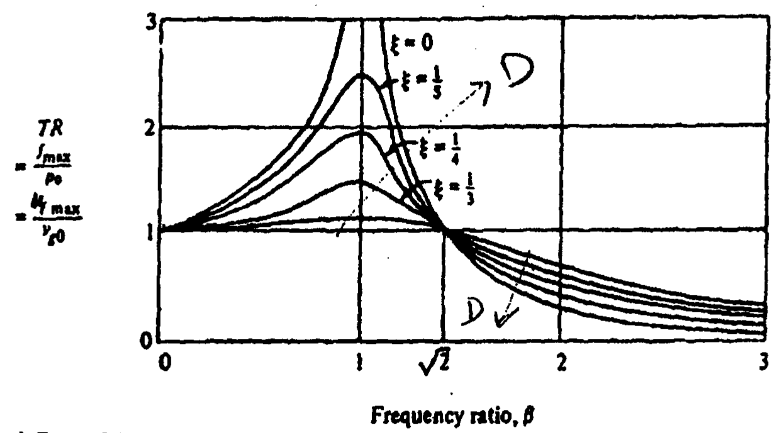


Figure 4.5 Vibration-transmissibility ratio (applied load or displacement).

The second type of situation in which vibration isolation is important is illustrated in Figure 4.4b. The mass m to be isolated is supported by a spring-damper system on a foundation slab which is subjected to harmonic vertical motions. The displacement of the mass relative to the base then is given by (See subchapter 4.3)

$$\mathbf{u(t) = u_{go}\beta^2 MF \sin(\Omega t - \theta)} \tag{4.28}$$

However, when the motion of the base is added vectorially it can be shown that the total motion of the mass is given by

$$\mathbf{u^t(t) = u_{go}\sqrt{1 + (2D\beta)^2}\,MF \sin(\Omega t - \theta)} \tag{4.29}$$

in which the phase angle θ is of no particular interest. Thus, if the transmissibility in this situation is defined as the ratio of the amplitude of motion of the mass to the base-motion amplitude, it can be seen that the expression for transmissibility is the same as that given by Equation 4.27. This can be expressed mathematically as

$$\mathbf{TR} = \frac{\mathbf{u^t_{max}}}{\mathbf{u_{go}}} = \mathbf{MF}\sqrt{\mathbf{1 + (2D\beta)^2}} \tag{4.30}$$

Figure 4.5 serves to define the transmissibility for both basic SDOF isolation situation.

For the design of a vibration-isolation system, it is convenient to express the behavior of the system in terms of its isolation effectiveness rather than the transmissibility, where the effectiveness is defined as 1-**TR**. It is evident from Figure 4.5 that the isolation mounting should have very little damping. Thus it ia acceptable to use the transmissibility expression for zero damping

$$\mathbf{TR} = \frac{1}{\beta^2 - 1} \tag{4.31a}$$

$$1 - \mathbf{TR} = \frac{\beta^2 - 2}{\beta^2 - 1} \tag{4.31b}$$

in which it is understood that $\beta \geq \sqrt{2}$. Finally it may be noted that

$$1 - \mathbf{TR} = \frac{\left(\frac{\Omega}{\omega}\right)^2 - 2}{\left(\frac{\Omega}{\omega}\right)^2 - 1}$$

or

$$\left(\frac{\Omega}{\omega}\right)^2 = \frac{2-(1-\mathbf{TR})}{1-(1-\mathbf{TR})}$$

$$\frac{\Omega^2}{k/m} = \frac{2-(1-\mathbf{TR})}{1-(1-\mathbf{TR})}$$

$$\frac{\Omega^2}{k.g/w} = \frac{2-(1-\mathbf{TR})}{1-(1-\mathbf{TR})}$$

$$\frac{\Omega^2}{g/\Delta_{st}} = \frac{2-(1-\mathbf{TR})}{1-(1-\mathbf{TR})}$$

$$\Omega = 19.65\sqrt{\frac{1}{\Delta_{st}}\frac{2-(1-\mathbf{TR})}{1-(1-\mathbf{TR})}} \qquad \sqrt{g} = \sqrt{386} = 19.65 \tag{4.32}$$

Equation 4.32 indicates that the effectiveness of the mounting system can be expressed in terms of the frequency of the input motion Ω and the static deflection value $\Delta_{st}=w/k$ (in inches). Δ_{st} is the deflection that the weight of the system to be isolated will produce on the vibration mounting system.

Figure 4.6 presents Equation 4.32 using $\bar{f} = \Omega/2\pi$. Knowing the frequency of the impressed vibration $\bar{f}$, one can determine directly from this graph the support-pad deflection Δ_{st} required to achieve any desired level of vibration isolation, assuming that the isolators have little damping. It also is apparent in the graph that the isolation system will have a harmful effect if it is too stiff (i.e. for the same frequency $\bar{f}$ decreasing Δ_{st} will result decreasing of isolation effectiveness).

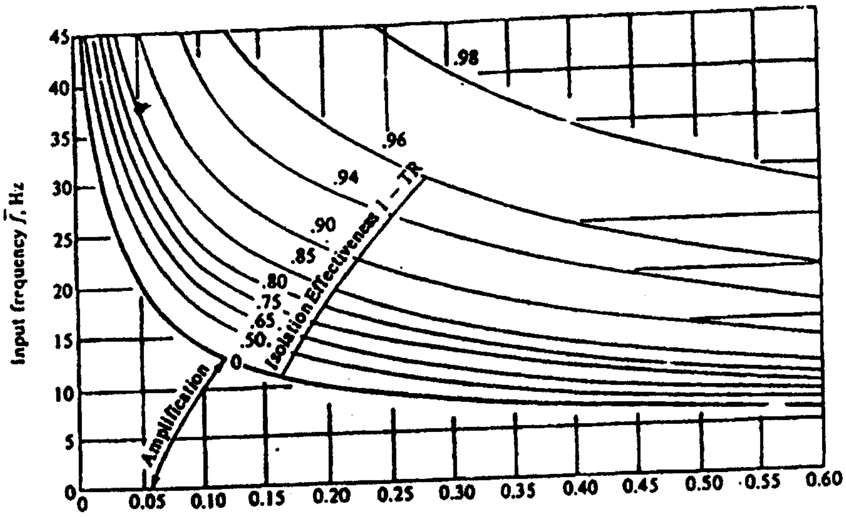


Figure 4.6 Vibration-isolation design chart.

Example:
A reciprocating machine weighing 20,000 1b is known to develop vertically oriented harmonic forces having amplitude of 500 1b at its operating speed of 40 Hz. In order to limit the vibrations excited in the building in which this machine is to be installed, it is to be supported by a spring at each corner of its rectangular base. The designer wants to know what support spring stiffness will be required to limit to 80 1b the total harmonic force transmitted from the machine to the building.

The transmissibility in this case is 80/500 = 0.16; hence by Equation 4.31a

$$\frac{1}{\mathbf{Tr}} = \left|\beta^2 - 1\right| = 6.25$$

from which

$$\beta^2 = 7.25 = \frac{\Omega^2 W}{k.g}$$

Solving for the total spring stiffness gives

$$k = \frac{\Omega^2 W}{7.25g} = 451 \times 10^3 \qquad 1b/in$$

Thus the stiffness of each of the four support spring is

$$k/4 = {}^{451}\!/_{4} = 113 \qquad kips/in$$

It is of interest to note that the static deflection caused by the weight of the machine on these spring supports is

$$\Delta_{st} = {}^{20}\!/_{451} = 0.0444 \text{ in}$$

4.6 EVALUATION OF DAMPING IN SDOF SYSTEMS

In the foregoing discussion of response analysis in SDOF structures, it has been assumed that the physical properties of the system (mass, stiffness, and damping) are known. In most cases, the structural mass and stiffness can be evaluated rather easily, either by simple physical considerations or by generalized expressions such as Equations 2.29a and 2.29c. On the other hand, the basic energy-loss mechanisms in practical structures are seldom fully understood; consequently it usually is not feasible to determine the damping coefficient by means of the corresponding generalized damping expression. For this reason, the damping in most structural systems must be evaluated directly from experimental methods. A brief survey of the principal procedures for evaluating damping from experimental measurements follows.

Free-Vibration Decay

Probably the simplest and most frequently used experimental method is measurement of the decay of free vibrations, as mentioned in Chapter 3. When a system has been set into free vibration by any means, the damping ratio can be determined from the ratio of two displacement amplitudes measured at an interval of **M** cycles. Thus if y_n is the amplitude of vibration at any time and y_{n+M} is the amplitude **M** cycles later, the damping ratio is given by

$$D = \frac{\delta_M}{2\pi M(\omega / \omega')} \doteq \frac{\delta_M}{2\pi M} \qquad (4.33)$$

where $\delta_M = \ln(y_n/y_{n+M})$ represents the logarithmic decrement and ω and ω' are the undamped and damped frequencies, respectively. In most practical structures, the damping ratio is less than 0.2, so that the approximate form of Equation 4.33, based on neglecting the change of frequency due to damping, is sufficiently accurate (the error in **D** is less than 2%). A major advantage of this free-vibration method is that equipment and instrumentation requirements are minimal; the vibrations can be initiated by any convenient method, and only the relative displacement amplitudes need to be measured.

Resonant Amplification

The other principal techniques for evaluating damping are based on observations of steady-state harmonic response behavior and thus require a means of applying harmonic excitations to the structure at prescribed frequencies and amplitudes. With such equipment the frequency-response curve for the structure can be constructed by applying a harmonic load $P_o \sin \Omega t$ at a closely spaced sequence of frequencies which span the resonance frequency and plotting the resulting displacement amplitudes as a function of the applied frequencies. A typical frequency-response curve for a moderately damped structure is shown in Figure 4.7.

The dynamic magnification factor for any given frequency is the ratio of the response amplitude at that frequency to the zero-frequency (static) response. It was shown earlier, that the damping ratio is closely related to the dynamic magnification factor at resonance. When the static-response and the resonant-response amplitude are denoted by ρ_o and $\rho_{\beta=1}$, respectively, the damping ratio is given by

$$D = \frac{1}{2}\frac{\rho_o}{\rho_{\beta=1}} \qquad (4.34)$$

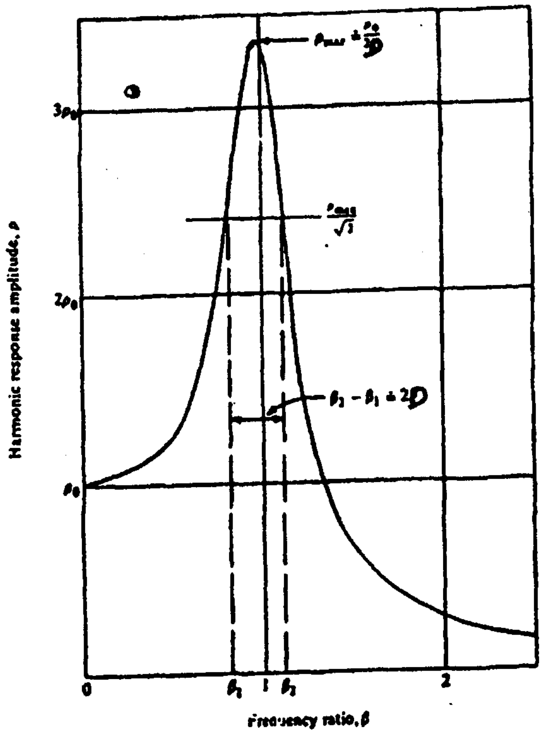


Figure 4.7 Frequency-response curve for moderately damped system.

In practice, however, it is difficult to apply the exact resonance frequency, and it is more convenient to determine the maximum response amplitude ρ_{max} which occurs at a slightly lower frequency. In this case, it is evident from Equation 4.18 that the damping ration may be evaluated as follows:

$$D = \frac{1}{2}\frac{\rho_o}{\rho_{max}}\frac{\omega}{\omega'} \doteq \frac{1}{2}\frac{\rho_o}{\rho_{max}} \tag{4.35}$$

The error involved in the simpler expression again results from neglecting the difference between the damped and undamped frequencies and is unimportant in ordinary structures.

This method of damping analysis requires only simple instrumentation, capable of measuring relative displacement amplitudes; however, the evaluation of the static displacement may present a problem because many types of loading systems cannot be operated at zero frequency.

Half-power (Bandwidth) Method

It is evident from the general harmonic-response expression [Equation 4.17a] that the shape of the entire frequency-response wave is controlled by the amount of damping in the system; therefore, it is possible to derive the damping ratio from many different properties of the curve. One of the most convenient of these is the bandwidth, or half-power, method, in which the damping ratio is determined from the frequencies at which the response is reduced to $(1/\sqrt{2})\rho_{\beta=1}$, that is, at the frequencies for which the power input is half the input at resonance.

The values of these half-power frequencies can be determined by setting the response amplitude in Equation 4.17a equal to $1/\sqrt{2}$ times the resonant amplitude derived from Equation 4.34, that is,

$$\frac{1}{\sqrt{2}}\frac{\rho_o}{2D}=\rho_o\left[\frac{1}{(1-\beta^2)^2+(2D\beta^2)}\right]^{1/2}=\rho_o MF$$

or, squaring both sides,

$$\frac{1}{8D^2}=\frac{1}{(1-\beta^2)^2+(2D\beta^2)}$$

solving for the frequency ratio then gives

$$\beta^2=1-2D^2\pm 2D\sqrt{1+D^2}$$

from which (neglecting D^2 in the square-root term) the two half-power frequencies are

$$\beta_1^2 \doteq 1-2D-2D^2 \qquad \beta_1=1-D-D^2$$
$$\beta_2^2 \doteq 1+2D-2D^2 \qquad \beta_2=1+D-D^2$$

Hence, the damping ratio is given by half the difference between these half-power frequencies:

$$D=\tfrac{1}{2}(\beta_2-\beta_1)$$

This method of evaluating the damping ratio also is illustrated with the typical frequency-response curve of Figure 4.7. A horizontal line has been drawn across the curve at $1/\sqrt{2}$ times the resonant-response value; the difference between the frequencies at which this line intersects the response curve is equal to twice the damping ratio. It is evident that this technique avoids the need for the static response; however, it does require that the response curve be plotted accurately in the half-power range and at resonance.

Example

Data from a frequency-response test of a **SDOF** system have been plotted in Figure 4.8. The pertinent data for evaluating the damping ratio are also shown. The sequence of steps in the analysis after the curve was plotted were as follows:

1. Determine peak response = 5.67×10^{-2} in
2. Construct line at peak $/\sqrt{2}=4.01\times10^{-2}$ in
3. Determine the two frequencies at which this line cuts the response curve: f_1=19.55, f_2=20.42
4. The damping ratio then is given by $D=\frac{\Delta\beta}{2}=\frac{f_2-f_1}{f_2+f_1}=2.18\%$

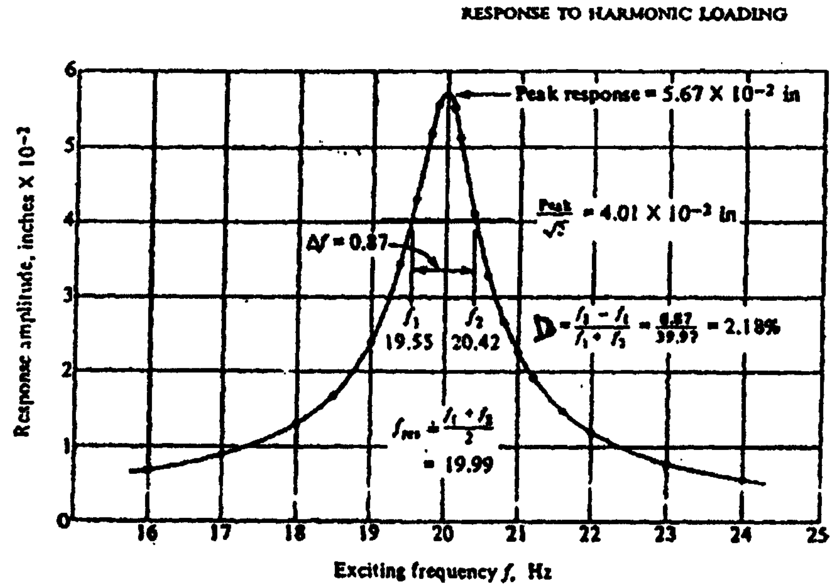


Figure 4.8 Frequency-response experiment to determine damping ratio.

Energy Loss per Cycle (Resonance Testing)

If instrumentation is available to measure the phase relationship between the input force and the resulting displacements, the damping can be evaluated from tests run only at resonance and there is no need to construct the frequency-response curve. The procedure involves establishing resonance by adjusting the input frequency until the response is 90° out of phase with the applied loading. Then the applied load is exactly balanced by the damping force, so that if the relationship between the applied load and the resulting displacements is plotted for one loading cycle as shown in Figure 4.9, the result can be interpreted as the damping-force-displacement diagram.

If the structure has linear viscous damping, the curve will be an ellipse, as shown by the dashed line in Figure 4.9. In this case, the damping coefficient can be determined directly from the ratio of the maximum damping force to the maximum velocity:

$$c = \frac{f_{D,max}}{\dot{y}_{max}} = \frac{P_o}{\omega \rho} \tag{4.35}$$

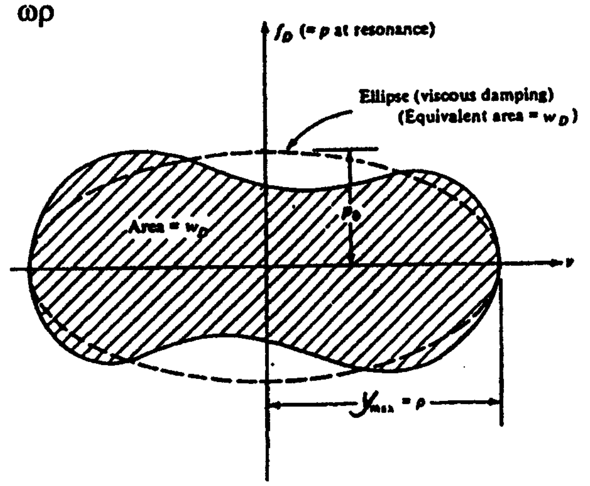


Figure 4.9 Actual and equivalent damping energy per cycle.

Where it is noted that the maximum velocity is given by the product of frequency and displacement amplitude. If damping is not linear viscous, the shape of the force-displacement diagram will not be elliptical; a curve like the solid line in Figure 4.9 might have been obtained, for example. In this case, an equivalent viscous damping coefficient can be defined which would cause the same energy loss per cycle as in the observed force-displacement diagram. In other words, the equivalent viscous damper is associated with the elliptical force-displacement diagram having the same area and maximum displacements as the actual force-displacement diagram. In this sense, the dashed-line curve in Figure 4.9 is equivalent to the solid-line curve. Then the equivalent applied-force amplitude is given by

$$P_o = \frac{w_D}{\pi\rho}$$

where w_D is the area under the force-displacement diagram, i.e., energy loss per cycle. Substituting this into Equation 4.35 leads to an expression for the equivalent viscous-damping coefficient in terms of the energy loss per cycle:

$$c_{eq} = \frac{P_o}{\omega\rho} = \frac{w_D}{\pi\omega\rho^2} \tag{4.36}$$

Note:

if force $P(t)=P_o \sin\Omega t$

then $m\ddot{y} + c\dot{y} + ky = P_0 \sin\Omega t$ assume $y = \rho\sin(\Omega t - \theta)$

then $-m\rho\Omega^2 \sin(\Omega t - \theta) + c\rho\Omega\cos(\Omega t - \theta) + k\rho\sin(\Omega t - \theta) = P_0 \sin\Omega t$ (a)

• static condition when $\frac{\Omega}{\omega} = 0$ then $\theta = 0$

$\therefore 0 + 0 + k\rho\sin\Omega t = P_0 \sin\Omega t$, $k\rho = P_o$

i.e. elastic force = amplitude of force

• at resonance $\Omega = \omega$ then $\theta = -\pi/2$

$-m\rho\omega^2 \cos\omega t - c\rho\omega\sin\omega t + k\rho\cos\omega t = P_0 \sin\omega t$

$-c\rho\omega\sin\omega t = P_0 \sin\omega t$, $-c\rho\omega = P_0$

$\therefore$ at resonance damping force = amplitude of force

• very large excitation frequency when $\frac{\Omega}{\omega} >>> 1$ then $\theta = -\pi$,

multiply Eq. 'a' by ω^2/Ω^2 then

$$-m\rho\omega^2 \sin(\Omega t + \pi) + c\rho\Omega\frac{\omega^2}{\Omega^2}\cos(\Omega t + \pi) + k\rho\frac{\omega^2}{\Omega^2}\sin(\Omega t + \pi) = P_0 \frac{\omega^2}{\Omega^2}\sin\Omega t$$

$$\therefore -m\rho\sin\Omega t = \frac{P_0}{\Omega^2}\sin\Omega t$$

$$m\rho\Omega^2 = P_0$$

i.e. inertia force = amplitude of force

In most cases, it is more convenient to define the damping in terms of the critical damping ratio than as a damping coefficient. For this purpose, it is necessary to define also a measure of the critical damping coefficient of the structure, and this can be expressed in terms of the mass and frequency [as in Equation 3.32] or in the alternate form involving stiffness and frequency:

$$c_{cr} = 2m\omega = \frac{2k\omega}{\omega^2} = \frac{k}{\omega} \tag{4.37}$$

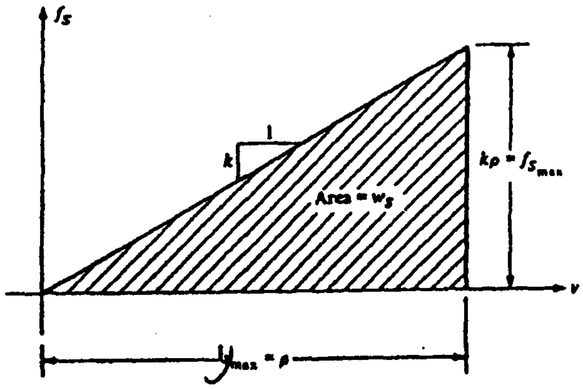


Figure 4.10 Elastic stiffness and strain energy.

This latter expression is more convenient here because the stiffness of the structure can be measured by the same instrumentation used to measure the damping-energy loss per cycle, merely by operating the system very slowly at essentially static conditions. The static-force-displacement diagram obtained in this way will be of the form shown in Fig 4.10. If the struct're is linearly elastic, and the stiffness is represented by the slope of the curve. Alternatively, the stiffness may be expressed by the area under the force-displacement diagram, w_s, as follows:

$$w_s = \frac{1}{2}(\rho)(\rho k) \qquad k = \frac{2w_s}{\rho^2} \tag{4.38}$$

Thus the damping ratio can be obtained by combining Equations 4.36 to 4.38:

$$D = \frac{c}{c_{cr}} = \frac{w_D}{4\pi w_s} \tag{4.39}$$

$$D = \frac{1}{4\pi}\frac{\text{energy}\cdot\text{loss}\cdot\text{per}\cdot\text{cycle}}{\text{maximum}\cdot\text{strain}\cdot\text{energy}}$$

The damping ratio defined by Equation 4.39 appears to be independent of frequency: it depends directly on the ratio of damping-energy loss per cycle to the strain energy stored at maximum displacement. However, for any given viscous-damping mechanism, the energy loss in the system will be proportional to the harmonic frequency, and thus the damping ratio will be too. Alternatively, when the damping ratio has been evaluated from a resonance test, the corresponding viscous-damping coefficient obtained by substituting Equation 4.36 into 4.39 is inversely proportional to the frequency:

$$c_{eq} = Dc_{cr} = D\frac{2k}{\omega} = D\frac{4w_s}{\omega\rho^2} \tag{4.40}$$

which again demonstrates the frequency dependence of the viscous-damping behavior as indicated on Figure 4.11

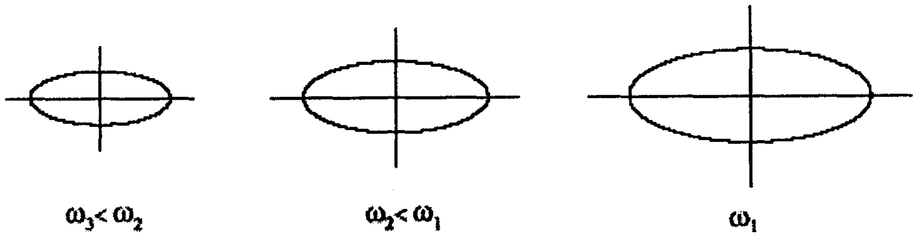


Figure 4.11 f_D increases with the increase of ω.

Hysteretic Damping

Although the viscous-damping mechanism leads to a convenient form of the structural equation of motion, the results of experiments seldom correspond closely with this type of energy-loss behavior. In many practical cases, the equivalent viscous-damper concept defined in terms of energy loss per cycle provides a reasonable approximation of experimental results. However, the essential frequency dependence of the viscous mechanism mentioned above is at variance with a great deal of test evidence, much of which indicates that damping forces are nearly independent of the frequency.

A mathematical model which has this property of frequency independence is provided by the concept of *hysteretic* damping, which may be defined as damping force in phase with the velocity but proportional to the displacements. This force-displacement relationship may be expressed as

$$f_D = Dk|y|\frac{\dot{y}}{|\dot{y}|} \tag{4.41}$$

where, D is the hysteretic-damping coefficient which defines the damping forces as a fraction of the elastic-stiffness forces. The force-displacement diagram for hysteretic damping during a cycle of harmonic displacement is depicted in Figure 4.12. It will be noted that the damping resistance is similar to the linear elastic forces during displacements of increasing magnitudes but that the sense of the damping force reverses when the displacements diminish. The hysteretic energy loss per cycle given by this mechanism is

$$w_D = 4(\frac{1}{2}\rho Dk\rho) = 2Dk\rho^2 \tag{4.42}$$

If this hysteretic energy loss is assumed to be represented by an equivalent viscous damper, the equivalent viscous-damping ratio still is given by Equation 4.39, In other words, Equation 4.39 can be used to express the damping ratio of a structure regardless of the actual internal-energy-loss mechanism. However, if the specific hysteretic-damping coefficient corresponding to a given test is to be determined, this may be expressed in terms of the damping ratio [by substituting Equations 4.42 and 4.38 into Equation 4.39] as

$$k = +2w_s / \rho^2$$

$$w_D = 2D\frac{2w_s}{\rho^2}\rho^2 = 4Dw_s$$

$$\therefore D = \frac{1}{4}\frac{w_D}{w_s}$$

$$\therefore c_{eq} = \frac{f_D}{|y|} = Dk$$

Note

Viscous damping $\rightarrow$ damping force = $c_v\dot{y} = c_v\omega y$

Hysteretic damping $\rightarrow$ damping force = $ic_h.y$

(i indicates out of phase)

$\therefore$ if hysteretic damping is replaced by equivalent viscous damping

$$\therefore c_{eq} = \frac{c_h}{\omega}$$

Proof

Viscous $\quad ky + c_v\dot{y} = P_o e^{i\omega t}$ (1)

hysteretic $\quad ky + ic_h y = P_o e^{i\omega t}$ (2)

assume solution $y = y_o e^{i\omega t}$

$\therefore$ Eq. (1) $\quad \rightarrow ky_o e^{i\omega t} + i\omega c_v y_o e^{i\omega t} = P_o e^{i\omega t}$

$$\therefore (k + i\omega c_v)y_o = P_o$$

Thus it is clear that the hysteretic-damping coefficient is independent of the frequency at which the test was run, in contrast with the frequency dependence of the viscous-damping coefficient shown by Equation 4.40.

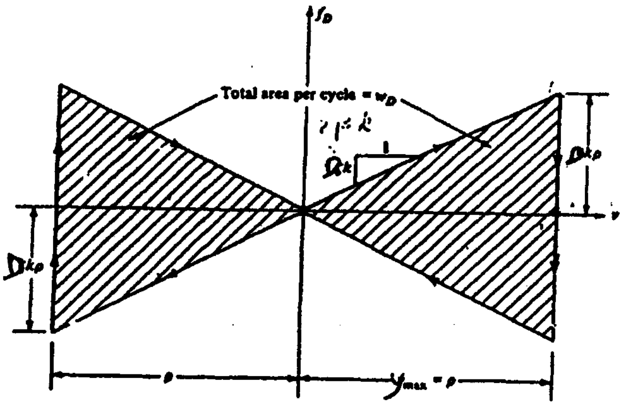


Figure 4.12 Hystereretic damping force versus displacement.

PROBLEMS

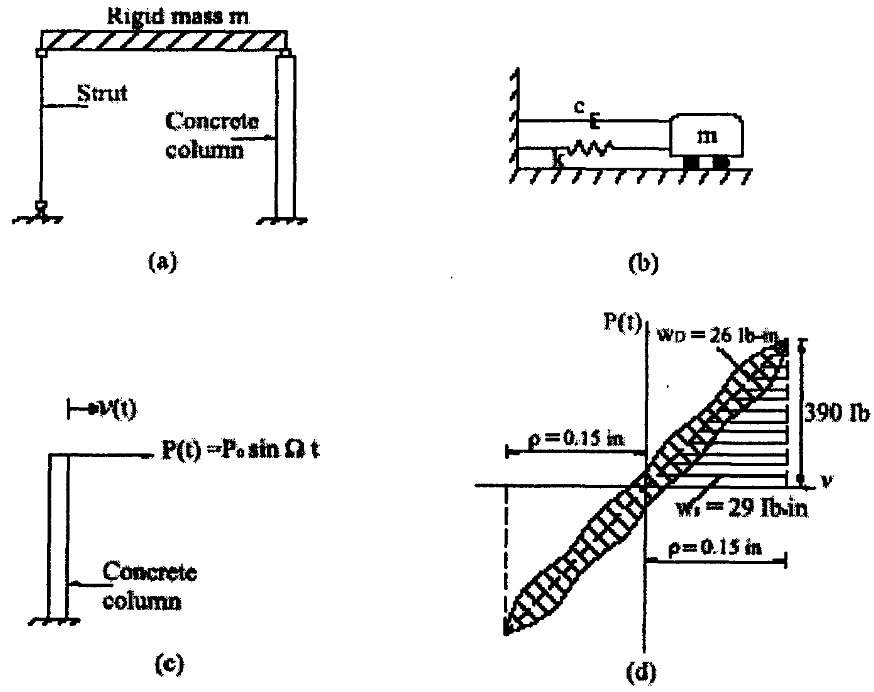
4.5 A control console containing delicate instrumentation is to be located on the floor of a test laboratory where it has been determined that the floor slab is vibrating vertically with an amplitude of 0.03 in at 20 Hz. If the weight of the console is 800 1b, determine the stiffness of the vibration isolation system required to reduce the vertical-motion amplitude of the console to 0.005 in.

4.6 A sieving machine weighs 6,500 1b and when operating at full capacity it exerts a harmonic force on its supports of 700 1b amplitude at 12 Hz. After mounting the machine on spring-type vibration isolators, it was found that the harmonic force exerted on the supports had been reduced to a 50-1b amplitude. Determine the spring stiffness **k** of the isolation system.

4.7 The structure of Figure p4-1a can be idealized by the equivalent system of Figure p-1b. In order to determine the values of **c** and **k** for this mathematical model, the concrete column was subjected to a harmonic load test as shown in Figure P4-1c. When operating at a tst frequency of Ω=10 rads/s, the fore-deflection (hysteresis) curve of Figure P4-1d was obtained. From this data:
(a) determine the stiffness **k**.
(b) assuming a viscous damping mechanism, determine the apparent viscous damping ratio **D** and damping coefficient **c**.
(c) assuming a hysteretic damping mechanism, determine the apparent hysteretic damping factor **D**.

4.8 Suppose that the test of problem 4.7 were repeated, using a test frequency Ω=20 rads/s, and that the fore-deflection curve (Figure P4-1d) was found to be unchanged. In this case:
(a) determine the apparent viscous damping values **D** and **c**.
(b) determine the apparent hysteretic damping factor **D**.
(c) Based on these two (Ω=10 and Ω=20 rad/s), which type of damping mechanism appears more reasonable-viscous or hysteretic?

4.9 If the damping of the system of problem 4.7 actually were provided by a viscous damper as indicated in Figure P4-1b, what would be the value of w_D obtained in a test performed at Ω =20 rads/s? (use Eq. 4.36)



Rigid mass m
Strut
Concrete column
(a)
c
k
m
(b)
v(t)
P(t) = P₀ sin Ω t
Concrete column
(c)
P(t)
w_D = 26 lb-in
390 lb
ρ = 0.15 in
v
w_s = 29 lb-in
ρ = 0.15 in
(d)

Figure P4.1

CHAPTER (5)

RESPONSE TO GENERAL DYNAMIC LOADING

Real structures are often subjected to loads which are not periodic. Such arbitrary general loading must be handled in a special way, as described in this chapter.

5.1 TRANSIENT LOADING AND DUHAMEL'S INTEGRAL

Consider the arbitrary transient load **P(t)** shown in Figure 5.1. The load has a limited duration t_p. The response of such load can be calculated by replacing the continuous load by a set of rectangular impulses **P(τ)dτ** of very short duration **dτ** and calculate the response for each of these impulses. Then get the total response by summing all tributary responses.

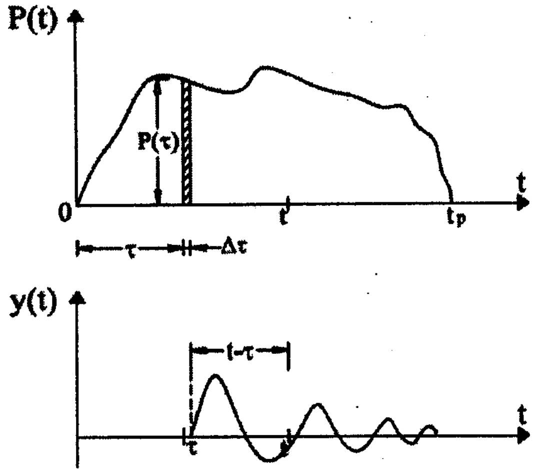


Figure 5.1 General forcing function as impulsive loading and its response.

The rectangular impulse acts at time τ while the response is to be calculated at time **t** (**t > τ**). This impulse acting on mass **m** produces a change of velocity **ΔV** which can be determined from the theorem of momentum

Momentum of the mass = impulse of the force

$$m\,\Delta V = P(\tau)\,d\tau \tag{5.1a}$$

or

$$\Delta V = \frac{P(\tau)d\tau}{m} \tag{5.1b}$$

Then this change of velocity is treated as if it were an initial velocity at time τ. The response of the mass at time **t** is the free vibration response due to initial velocity given by Equation 3.42

$$dy(t) = \frac{\Delta V}{\omega`} e^{-\omega D(t-\tau)} \sin \omega`(t-\tau)$$

$$= \frac{P(\tau)d\tau}{m\omega`} e^{-\omega D(t-\tau)} \sin \omega`(t-\tau) \tag{5.2}$$

Summation or integration of the contributions **dy(t)** from all impulses **P(τ)dτ** from time **t**= 0 to the time of observation, **t**, then response to transient load is

For $t < t_p$
$$y = \frac{1}{m\omega'}\int_0^t P(\tau)e^{-\omega D(t-\tau)}\sin\omega'(t-\tau)d\tau \qquad (5.3)$$

Equation 5.3 is known as the ***Duhamel integral.*** It may be used to evaluate the response of a SDOF system to any form of dynamic loading **P(t)**.For simple forcing function, it is possible to obtain the explicit integration of Equation 5.3. With very irregular time histories, the evaluation will have to be performed numerically.

After the transient load expires, i.e. for $t > t_p$, motion becomes free vibration started by initial displacement $y(t_p)$ and initial velocity $\dot{y}(tp)$ and may be expressed as follows (see Equation 3.37).

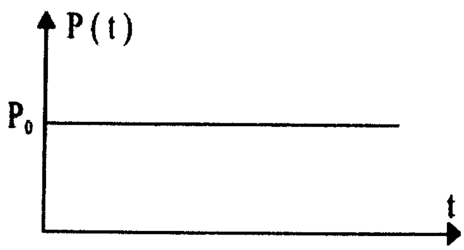
For $t > t_p$
$$y(t) = e^{-\omega D(t-t_p)}[y(t_p)\cos\omega'(t-t_p) + \frac{\dot{y}(t_p) + y(t_p)\omega D}{\omega'}\sin\omega'(t-t_p)] \qquad (5.4)$$

in which $y(t_p)$ and $\dot{y}(t_p)$ are obtained from Equation 5.3 for $t = t_p$.

Example 1

Response of undamped SDOF system to suddenly applied static force $P(t) = P_o$.
From Equation 5.3, the response is

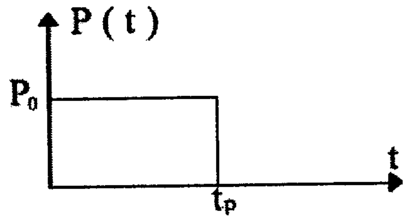
$$\begin{aligned}
y(t) &= \frac{P_o}{m\omega}\int_0^t \sin\omega(t-\tau)d\tau \\
&= \frac{P_o}{m\omega^2}[\cos\omega(t-\tau]_0^t \\
&= \frac{P_o}{k}[1-\cos\omega t] \\
&= y_{st}[1-\cos\omega t]
\end{aligned}$$



(5.5)

Which is the same response obtained in subchapter 4.4.

Example 2
Find the response of a SDOF system due to constant force P_o
Suddenly applied but only during a limited time duration t_p



a) for $t < t_p$ the solution is given Equation 5.5

for $t = t_p$ $\quad y(t_p) = y_{st}(1 - \cos\omega t_p)$

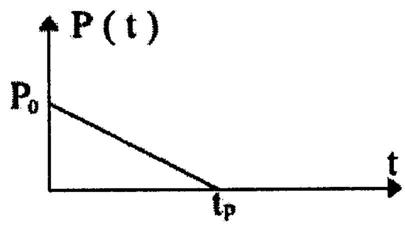
$\dot{y}(t_p) = y_{st}\,\omega \sin\omega t_p$

b) for $t > t_p$, apply Equation 3.15 or Equation 5.4 for free vibration, taking as the initial conditions the displacement and velocity at t_p, then

$$y(t) = y_{st}(1-\cos\omega t_p)\cos\omega(t - t_p) + y_{st}\sin\omega t_p \sin\omega(t - t_p)$$

or $$y(t) = y_{st}[\cos\omega(t - t_p) - \cos\omega t] \tag{5.6}$$

Example 3 ***Triangular impulse***
Find the response of a SDOF system to a force **P(t)** which has an initial value P_o and which decreases linearly to zero at time t_p



a) for $t < t_p$ $\quad P(t) = P_o\left(1 - \frac{t}{t_p}\right)$

From Equation 5.3, with $D = o$

$$y = \frac{t}{m\omega}\int_o^t P_o\left(1 - \frac{\tau}{t_p}\right)\sin\omega(t-\tau)\,d\tau$$

$$= \frac{P_o}{k}\left(\frac{\sin\omega t}{\omega t_p} - \cos\omega t - \frac{t}{t_p} + 1\right) \tag{5.7}$$

at $t = t_p$

$$y(t_p) = \frac{P_o}{k}\left(\frac{\sin\omega t_p}{\omega t_p} - \cos\omega t_p\right) \tag{5.8}$$

$$\dot{y}(t_p) = \frac{P_o}{k}\left(\omega\sin\omega t_p + \frac{\cos\omega t_p}{t_p} - \frac{1}{t_p}\right) \tag{5.9}$$

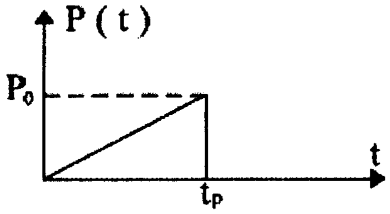
b) for $t > t_p$

replacing, t, in Equation 3.15 t by $t - t_p$ and $y(o)$ and $\dot{y}(o)$ respectively, by $y(t_p)$ and $\dot{y}(t_p)$, then

$$y = \frac{P_o}{k\omega t_p}\{\sin\omega t - \sin\omega(t - t_p)\} - \frac{P_o}{k}\cos\omega t \tag{5.10}$$

Problem Predict the undamped response to the triangular impulse shown
Note,

$$P(t) = P_o \frac{t}{t_p}$$



answer

$$y(t) = \frac{P_o}{kt_p}\left(t - \frac{\sin \omega t}{\omega}\right) \qquad 0 < t \le t_p$$

$$y(t) = \frac{P_o}{kt_p}\left(t_p \cos \omega(t - t_p) + \frac{1}{\omega}\sin \omega(t - t_p) - \frac{\sin \omega t}{\omega}\right) \qquad t > tp$$

5.2 RESPONSE TO EARTHQUAKE GROUND MOTION

The motion of the SDOF structure subjected to earthquake ground motion is governed by (Equation 2.2)

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g \qquad (2.2)$$

Since earthquake forces ($-m\ddot{u}_g$) are transient, then response to such loads can be evaluated using the Duhamel integral (Equation 5.3 with $P(\tau) = -m\ddot{u}_g(\tau)$ and neglecting the minus sign:

$$u(t) = \frac{1}{\omega`}\int_0^t \ddot{u}_g(\tau) e^{-\omega D(t-\tau)} \sin \omega`(t-\tau)d\tau$$

or $$u(t) = \frac{1}{\omega`}V(t) \qquad (5.11)$$

In which

$$V(t) = \int_0^t \ddot{u}_g(\tau) e^{-\omega D(t-\tau)} \sin \omega`(t-\tau)d\tau \qquad (5.12)$$

For a given time history $\ddot{u}_g(t)$, the integral $V(t)$ can be evaluated and the complete response time history $u(t)$ can be established.

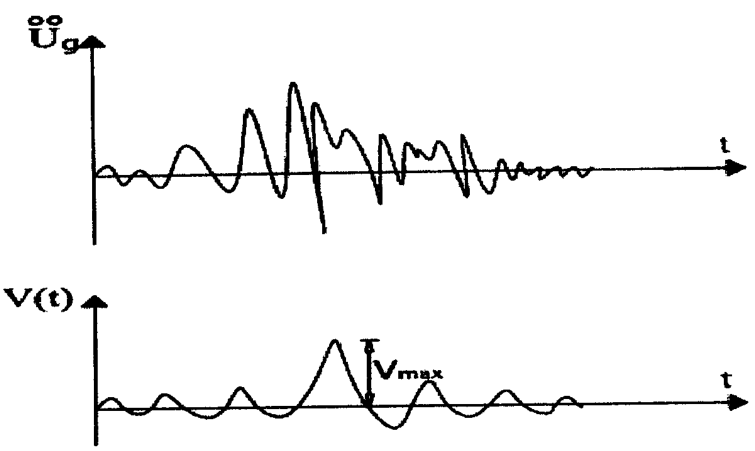


Figure 5.2 Earthquake ground motion and resulting structural response V(t)=ω`u(t)

However, for design purposes, it is often sufficient to know the maximum (peak) response. This peak value corresponds to the maximum value of **V(t)**.
As can be seen from Equation 5.12, the function **V(t)** depends only on the given ground motion $\ddot{u}_g(t)$, and the structural properties **ω** and **D**.

Thus for a given ground motion, the values of V_{max} can be calculated for various values of ω and **D** and collected in a graph called ***"Pseudo-velocity spectrum"*** which is the basis of seismic loading of buildings. This pseudo-velocity spectrum is usually calculated for a number of earthquake motions and then, the resulting spectra are smoothed and averaged and normalized to peak ground acceleration equal to a percentage of **g** as shown in Figure 5.3.

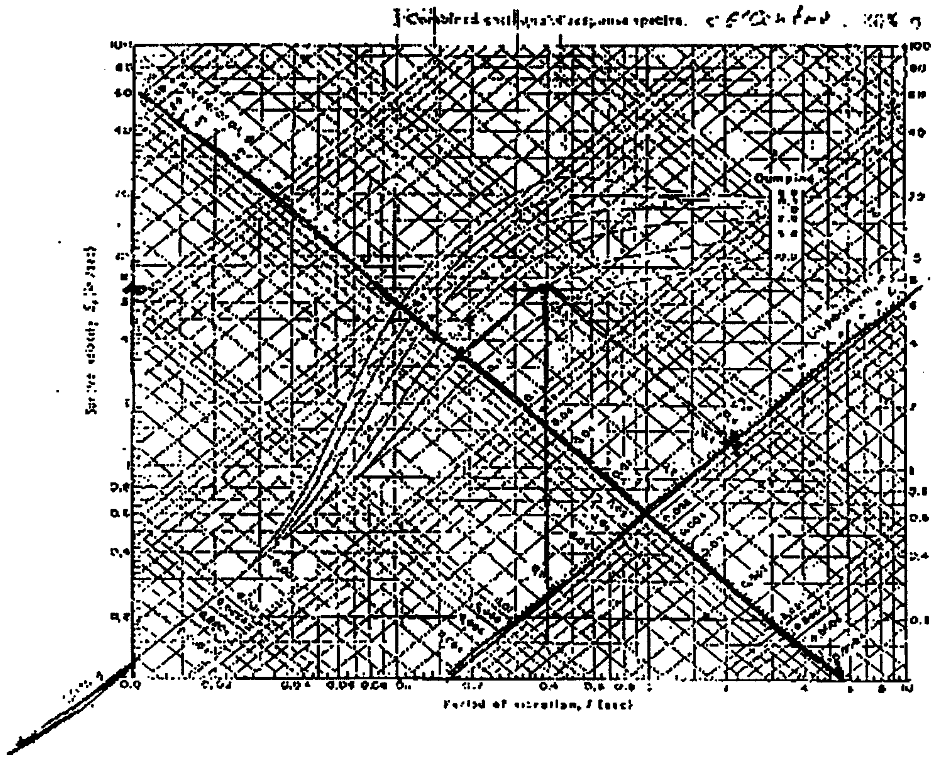


Figure 5.3 Smoothed averaged spectrum for. 2g maximum ground acceleration.

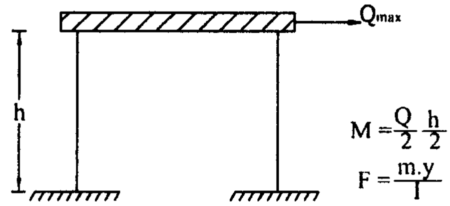
With V_{max} established, then using Equation 5.12 the maximum (spectral) displacement $S_d = S_v / \omega$ can be optioned. Finally, the approximate maximum acceleration is defined as $S_a = \omega . S_v$ (it is sufficient to replace ω` by ω). All values of S_v, S_d, S_a are presented in the same plot in Figure 5.3.

Then the maximum effective earthquake force, Q_{max}, is defined as

$$Q_{max} = m . S_a \tag{5.13}$$

Thus, the basic magnitudes used in the pseudo velocity approach are:

Approximate maximum velocity $V_{max} = S_v$
Approximate maximum displacement $S_d = S_v/\omega$
Approximate maximum acceleration $S_a = S_v . \omega$
Approximate maximum effective E.Q. Force $Q_{max} = m\ S_a$
Then Q_{max} is used as a static force in dimensioning the structure without any further consideration of the ground motion



Problem:
Analyze the one story shear building described in problem 3.3 for the effect of an earthquake given by the peak acceleration of 30% **g** and pseudo velocity spectrum in Figure 5.3. Assume both columns with hinges at the top of the columns and without hinges. Also assume damping ratio 2%. Calculate:

a) The maximum response, i.e. S_d, S_v, S_a,
b) The effective earthquake force and the stress in the foot of the column.

5.3 DESCRIBING SEISMICITY

The seismicity of a certain region can be described using few items:

Focus is the point in the earth's crust where the rupture occurs and a seismic wave originates. The point lying on the ground surface in the vertical projection of the focus is called the 'epicenter"

Intensity of an earthquake is a measure of the local destructiveness of an earthquake. One such intensity scale is the Mercalli Intensity Scale shown in the table. Because local destructiveness changes with distance from the epicenter, intensity cannot describe the seismic event.

Magnitude of an earthquake is a measure of the energy released. It is expressed in Richter scale and is defined as

$$\mathbf{M = \log_{10} (A/A_o) = \log A - \log A_o} \tag{5.14}$$

In which (Figure 5.4)
A = amplitude of ground acceleration recorded in an earthquake.
A_o = minimum amplitude still registered in a region (typical value is 1 mμ or 0.001 mm/s^2)

Structural damage usually starts at magnitude of 5. The empirical relation between magnitude M and energy released W in ergs is:

$$\mathbf{\log_{10} W = 11.8 + 1.5\, M} \tag{5.15}$$

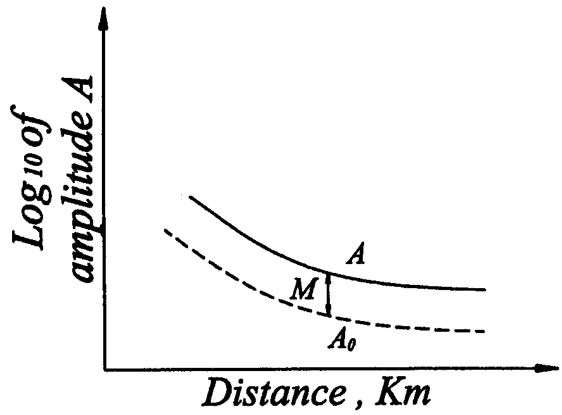


Figure 5.4 Description of earthquake magnitude.

Modified Mercalli Intensity Scale of 1931

I	Not felt except by very few under especially favorable circumstances
II	Felt only by a few persons at rest, especially on upper floors of buildings. Delicately suspended objects may swing.
III	Felt quite noticeably indoors, especially on upper floors of buildings, but many, people do not recognize it as an earthquake. Standing motor cars may rock slightly. Vibrations like passing of truck. Duration estimated
IV	During the day felt indoors by many, outdoors by few. At night some awakened. Dishes, windows, doors disturbed; walls make creaking sound. Sensation like heavy truck striking building. Standing motor cars rocked noticeably.
V	Felt by nearly everyone; many awakened Some dishes, windows, etc., broken; a few instances of cracked plaster, unstable objects overturned. Disturbance of trees, poles and other tall objects sometimes noticed. Pendulum clocks may stop.
VI	Felt by all; many frightened and run outdoors. Some heavy furniture moved; a few instances of fallen plaster or damaged chimneys. Damage slight.
VII	Everybody runs outdoors. Damage negligible in buildings of good design and construction; slight to moderate in well-built ordinary structures; considerable in poorly built or badly designed structures, some chimneys broken. Noticed by persons driving motor cars.
VIII	Damage slight in-specially designed structures; considerable in ordinary substantial buildings with partial collapse; great in poorly built structures. Panel walls thrown out from structures. Fall of chimneys, factory stacks, columns, monuments, and walls. Heavy furniture overturned. Sand and mud ejected in small amounts. Changes in well water. Persons driving motor cars disturbed.
IX	Damage considerable in specially designed structures; well designed frame structures thrown out of plumb; great in substantial buildings, with partial collapse. Buildings shifted off foundations. Ground cracked conspicuously. Underground pipes broken.
X	Some well-built wooden structures destroyed; most masonry and frame structures destroyed with foundations, ground badly cracked. Rails bent. Landslides considerable from river banks and steep slopes. Shifted sand and mud. Water splashed (slopped) over banks.
XI	Few, if any (masonry), structures remain standing, Bridges destroyed. Broad fissures in ground. Underground pipe lines completely out of service. Earth slumps and land slips in soft ground. Rails bent greatly.
XII	Damage total. Waves seen on ground surfaces. Lines of sight and level distorted. Objects thrown upward into air.

CHAPTER (6)

MULTI-DEGREES-OF-FREEDOM SYSTEMS

If the physical properties of the system are such that its motion can be described by a single coordinate and no other motion is possible, then it actually is a SDOF system and the solution of the differential equation of motion provides the exact dynamic response. On the other hand, if the structures actually has more than one possible mode of displacement and it is reduced mathematically to a SDOF approximation by assuming its deformed shape, the solution of the equation of motion is only an approximation of the true dynamic behavior and it may be difficult to asses the reliability of the results obtained.

In general, the dynamic response of a structure cannot be described adequately by a SDOF model; usually the response includes time variations of the displacement shape as well as its amplitude. Such behavior can be described only in terms of more than one displacement coordinate (one degree of freedom). The MDOF system refers to both the finite element and lumped mass type of idealization.

In this chapter the development of the equation of motion of a general MDOF system is demonstrated first. Then the free and forced vibrations of MDOF system are described.

6.1 EQUATIONS OF MOTION

Equation of motion can be again obtained as equations of dynamic equilibrium of all acting forces. This can be illustrated by means of the following example of a damped shear building featuring both relative and absolute damping (Figure 6.1).

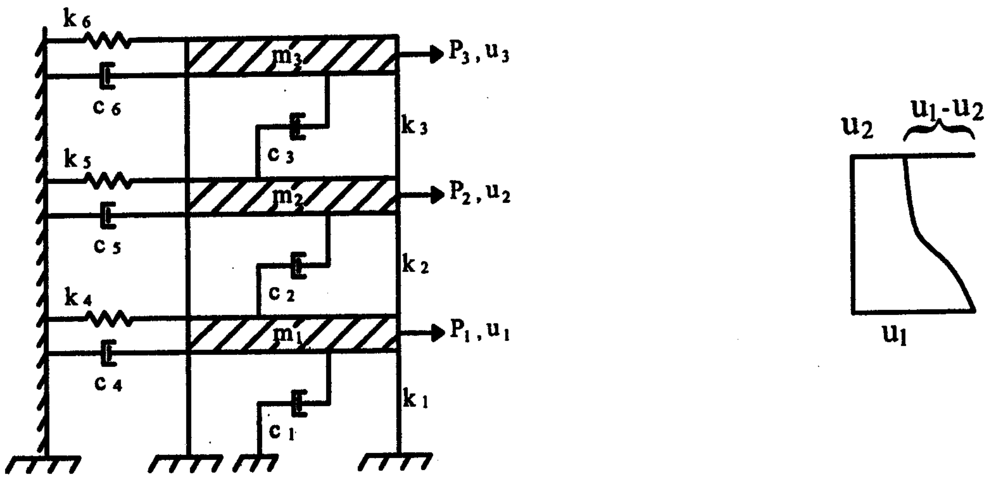


Figure 6.1 Shear Building.

Absolute Damping Force = c_i x absolute velocity
Relative Damping Force = c_i x relative velocity
Relative velocity = $\dot{u}_i$ **(piston)** $-\ \dot{u}_{i-1}$ **(cylinder)**
Stiffness constants of columns may be defined for a shear building by positive numbers

$$k_i = \frac{12EI_i}{\ell_i^3} \times N, \qquad \text{N= no of columns, etc.}$$

Alternatively stiffness and damping constants can be defined for joints as double subscripted magnitudes k_{ij} and c_{ij} in the same way as used in structural theory. (k_{ij} = force acting at mass i if there is a sole unit displacement at mass j, c_{ij} = the force at mass i required to produce a unit velocity at mass j)
Applying Newton's second law to individual masses and superimposing the effects of the individual displacements and velocities, the equations of equilibrium are:

For the first mass

$$m_1\ddot{u}_1 = -(k_1 + k_4 + k_2)u_1 + k_2u_2 - (c_1 + c_2 + c_4)\dot{u}_1 + c_2\dot{u}_2 + p_1$$

For the second mass

$$m_2\ddot{u}_2 = k_2u_1 - (k_2 + k_3 + k_5)u_2 + k_3u_3 - (c_2 + c_3 + c_5)\dot{u}_2 + c_3\dot{u}_3 + p_2$$

For the third mass

$$m_3\ddot{u}_3 = k_3u_2 - (k_3 + k_6)u_3 + c_3\dot{u}_2 - (c_3 + c_6)\dot{u}_3 + p_3$$

Using double subscripted stiffness and damping constants

$$m_i\ddot{u}_i + \sum_{r=1}^{n} k_{ir}u_r + \sum_{r=1}^{n} c_{ir}\dot{u}_r = p_i(t) \qquad \text{for } i = 1,2,\ldots,n.$$

This can be rearranged and written in matrix form

$$\lceil m \rfloor \{\ddot{u}\} + [c]\{\dot{u}\} + [k]\{u\} = \{p\} \tag{6.1}$$

in which

$$\lceil m \rfloor = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \qquad \{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \qquad \{p\} = \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix}$$

$$[c] = \begin{bmatrix} c_1 + c_2 + c_4 & -c_2 & 0 \\ -c_2 & c_2 + c_3 + c_5 & -c_3 \\ 0 & -c_3 & c_3 + c_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$[k] = \begin{bmatrix} k_1 + k_2 + k_4 & -k_2 & 0 \\ -k_2 & k_2 + k_3 + k_5 & -k_3 \\ 0 & -k_3 & k_3 + k_6 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

Notice that [c] and [k] are symmetric as $k_{ij} = k_{ji}$, $c_{ij} = c_{ji}$. The equations of the motion can be established directly by formulating the stiffness and mass matrices for the whole structure.

Problem: Derive matrix equations of motion and formulate the mass, stiffness and damping matrices for the structures shown in Figures 6.2a, b, and c

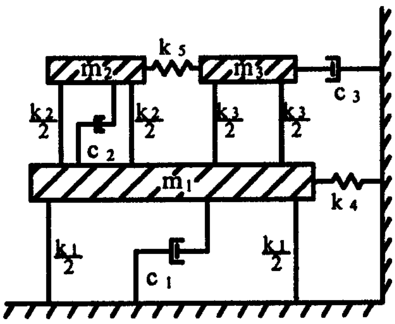


Figure 6.2a

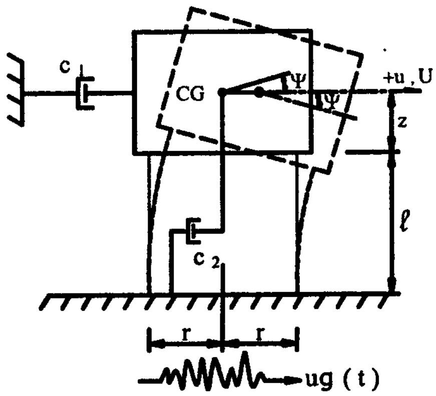


Figure 6.2b

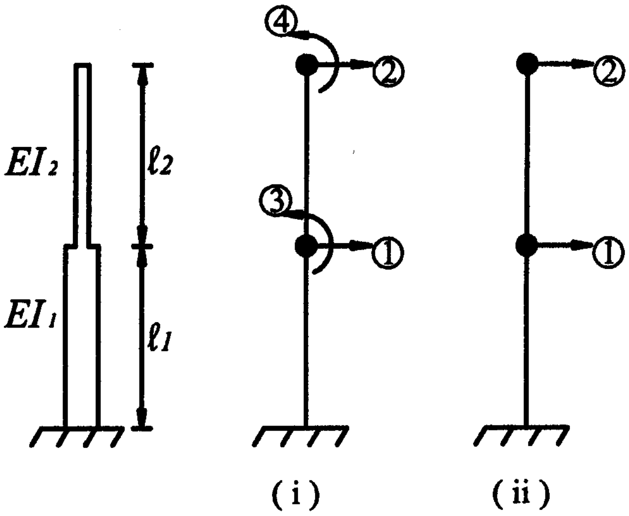
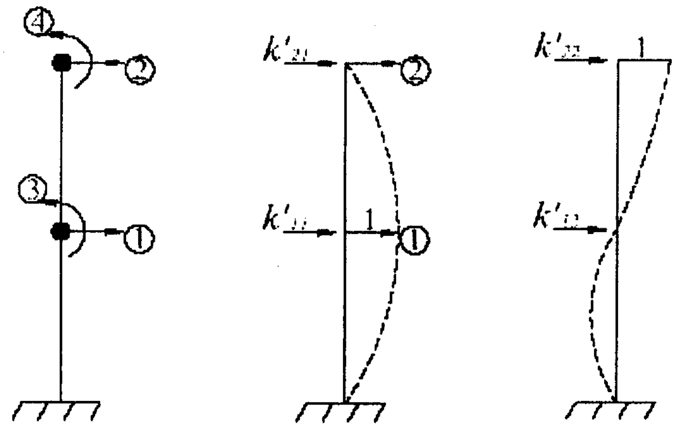


Figure 6.2 c: (i) formulate full stiffness matrix 4 × 4
(ii) condense the stiffness matrix to 2 × 2



$$k_{11} = \frac{12EI_1}{\ell_1^3} + \frac{12EI_2}{\ell_2^3}$$

$$k_{22} = \frac{12EI_2}{\ell_2^3}$$

$$k_{33} = \frac{4EI_1}{\ell_1} + \frac{4EI_2}{\ell_2}$$

$$k_{44} = \frac{4EI_2}{\ell_2}; \qquad k_{34} = \frac{2EI_2}{\ell_2}$$

$$k_{21} = \frac{-12EI_2}{\ell_2^3}$$

$$k_{23} = \frac{6EI_2}{\ell_2^2}; \qquad k_{24} = \frac{6EI_2}{\ell_2^2}$$

$$k_{31} = \frac{6EI_1}{\ell_1^2} - \frac{6EI_2}{\ell_2^2}$$

$$k_{41} = -\frac{6EI_2}{\ell_2^2}$$

[k] = Full matrix, 4 × 4:

$$\begin{Bmatrix} p_1 \\ \underline{p_2} \\ 0 \\ 0 \end{Bmatrix} = \left[\begin{array}{cc|cc} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ \hline k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{array}\right] \begin{Bmatrix} u_1 \\ \underline{u_2} \\ \psi_1 \\ \psi_2 \end{Bmatrix}$$

$$\begin{Bmatrix} P \\ \hline 0 \end{Bmatrix} = \left[\begin{array}{cc} A & B \\ \hline B^T & C \end{array}\right] \begin{Bmatrix} u \\ \hline \psi \end{Bmatrix}$$

$$\underset{2\times1}{\{\,0\,\}} = \underset{2\times2}{[\,B\,]^T}\underset{2\times1}{\{\,u\,\}} + \underset{2\times2}{[\,C\,]}\underset{2\times1}{\{\,\psi\,\}} \Rightarrow \qquad \{\psi\} = -[\,C\,]^{-1}[\,B\,]^T\{\,u\,\}$$

$$\begin{bmatrix} k`_{11} & k`_{12} \\ k`_{21} & k`_{22} \end{bmatrix} = [\,T\,]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underset{2\times2}{[\,k`\,]} = \text{condensed matrix}, 2\times2$$

$$\{P\} = [\,A\,]\{\,u\,\} - [B][C]^{-1}[B]^T\{\,u\,\} = \overbrace{\{A] - [B][C]^{-1}[B]^T\}}^{[T]}\{u\}$$

for k_{11}: $P_1 = k`_{11}$, $P_2 = k`_{21}$, $u_1 = 1, u_2 = 0$

for k_{22}: $P_1 = k`_{12}$, $P_2 = k`_{22}$, $u_1 = 0, u_2 = 1$

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = [k`]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \text{or in general } \{P\} = \underset{n\times n}{[k`]}[u]$$

n = number of masses

6.2 FREE UNDAMPED VIBRATION OF MDOF SYSTEM

The basic type of response is the free undamped vibration. The solution of the free undamped equations of motion yields the natural frequencies and mode shapes of vibration. For free undamped vibration equation 6.1 becomes

$$\lceil m \rfloor \{\ddot{u}\} + [k]\{u\} = \{0\} \tag{6.2}$$

in which the diagonal mass matrix is

$$\lceil m \rfloor = \begin{bmatrix} m_1 & 0 & 0 & \cdots \\ 0 & m_2 & 0 & \cdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & m_n \end{bmatrix}$$

The displacement vector (column matrix)

$$\{u(t)\} = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \ \vdots \\ u_n(t) \end{bmatrix} = [u_1(t)\ u_2(t) \ldots u_n(t)]^T$$

And the symmetrical stiffness matrix [k] is

$$[k] = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & & & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix}$$

where, $k_{ij} = k_{ji}$

For free undamped vibration, the particular solutions to equations 6.2 must describe harmonic motions as in one degree of freedom and hence

$$u_i(t) = u_i \sin \omega t$$

$$\ddot{u}_i(t) = -u_i \omega^2 \sin \omega t$$

or

$$\{u(t)\} = \{u\} \sin \omega t, \qquad \{\ddot{u}\} = -\omega^2 \{u\} \sin \omega t \tag{6.3}$$

in which {u} = the vector of amplitudes (independent of time). Substituting equation 6.3 into equation 6.1 yields

$$-\lceil m \rfloor \omega^2 \{u\} \sin \omega t + [k]\{u\} \sin \omega t = 0$$

or

$$([k] - \omega^2 \lceil m \rfloor)\{u\} = 0 \tag{6.4}$$

in which the column matrix of amplitudes

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = [u_1 u_2 \ldots u_n]^T$$

Equation 6.4 is a set of homogeneous algebraic equations for amplitudes $\mathbf{u_i}$ in which frequency ω is unknown. The problem of finding the unknown frequencies ω from Equation 6.4 is called the ***Eigen value problem.***

The solution of the ***Eigen value problem*** utilizes the basic properties of homogeneous algebraic equation which imply that the roots (unknowns) are nonzero (nontrivial) only if the determinant of the coefficients vanishes. Thus, from equation 6.4 the characteristic determinant $\Delta = 0$, i.e.

$$|\, [k] - \omega^2 \lceil m \rfloor \,| = 0 \tag{6.5}$$

For some types of numerical solutions, for example for the solution by means of computer, tnis condition may be somewhat rearranged. Introduce $\lambda = 1/\omega^2$ and mutiply equation 6.4 by $-\lambda$. Also premultiply by $[k]^{-1}$ realizing that

$$[k]^{-1}\,[k] = [I]$$

where, **[I]** is the identity matrix. Then equation 6.4 becomes

$$(-\lambda\,[I] + [k]^{-1}\,\lceil m \rfloor)\,\{u\} = 0$$

or

$$([k]^{-1}\,\lceil m \rfloor - \lambda[I]\,\{u\} = 0 \tag{6.6}$$

Again, a nontrivial solution exists only if the determinant of the coefficients vanishes, i.e.

$$\left|\, [k]^{-1}\,\lceil m \rfloor - \lambda[I] \,\right| = 0 \tag{6.7}$$

Finding values of λ for which equation 6.7 is satisfied represents the eigen value problem of the nonsymmetrical matrix

$$[k]^{-1}\,\lceil m \rfloor = [D] = \textbf{Dynamical Matrix}$$

After the **eigen values (natural frequencies)** have been established the ratios of the unknown displacements (called vibration modes) can be found from the homogeneous algebraic equations, equation 6.4. For each frequency substituted in equation 6.4 one mode is obtained. These modes are called **eigen vectors**.

When the characteristic equation 6.5 or 6.7 are worked out they yield an ordinary algebraic equation for the unknown frequency, ω; this equation is of the order **n** for ω^2. The frequency vector will be:

$$\omega = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{Bmatrix} \tag{6.8}$$

6.2.1 Analysis of Vibration Mode Shapes

When the frequencies of vibration have been determined from equation 6.5, the equation of motion [equation 6.4 may be expressed as

$$\tilde{\mathbf{E}}^{(n)} \ \mathbf{u} = 0 \tag{6.9}$$

in which

$$\tilde{\mathbf{E}}^{(n)} = [\mathbf{k}] - \omega_n^2 [\ulcorner \mathbf{m} \lrcorner] \tag{6.9a}$$

Thus $\tilde{\mathbf{E}}^{(n)}$ represents the matrix obtained by subtracting $\omega_n^2[\ulcorner \mathbf{m} \lrcorner]$ from the stiffness matrix; since it depends on the frequency, it is different for each mode. Equation 6.9 is satisfied identically because frequencies were evaluated from this condition; therefore the ***amplitude*** of the vibration is indeterminate. However, the ***shape*** of the vibrating system can be determined by solving for all displacements in terms of any one coordinate.
For this purpose it will be assumed that first element of the displacement vector has unit amplitude, that is,

$$\begin{bmatrix} u_{1n} \\ u_{2n} \\ u_{3n} \\ \vdots \\ u_{Nn} \end{bmatrix} = \begin{bmatrix} 1 \\ u_{2n} \\ u_{3n} \\ \vdots \\ u_{Nn} \end{bmatrix} \tag{6.10}$$

In expanded form, equation 6.9 may then be written

$$\left[\begin{array}{c|cccc} e_{11}^{(n)} & e_{12}^{(n)} & e_{13}^{(n)} & \dots & e_{1N}^{(n)} \\ \hline e_{21}^{(n)} & e_{22}^{(n)} & e_{23}^{(n)} & \dots & e_{2N}^{(n)} \\ e_{31}^{(n)} & e_{32}^{(n)} & e_{33}^{(n)} & \dots & e_{3N}^{(n)} \\ \cdots & \cdots & \cdots & & \cdots \\ e_{Nn}^{(n)} & e_{N2}^{(n)} & e_{N3}^{(n)} & \dots & e_{NN}^{(n)} \end{array}\right] \left[\begin{array}{c} 1 \\ \hline u_{2n} \\ u_{3n} \\ \cdot \\ u_{Nn} \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{array}\right] \tag{6.11}$$

in which partitioning is indicated to correspond with the as yet unknown displacement amplitudes. For convenience, equation 6.11 will be expressed symbolically as

$$\begin{bmatrix} e_{11}^{(n)} & \tilde{E}_{10}^{(n)} \\ \tilde{E}_{01}^{(n)} & \tilde{E}_{00}^{(n)} \end{bmatrix} \begin{bmatrix} 1 \\ u_{0n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.11a}$$

from which

$$\tilde{E}_{01}^{(n)} + \tilde{E}_{00}^{(n)} u_{0n} = 0 \tag{6.12}$$

as well as

$$e_{11}{}^{(n)} + \tilde{E}_{10}^{(n)} u_{0n} = 0 \tag{6.13}$$

Equation 6.12 can be solved simultaneously for the displacement amplitudes

$$u_{0n} = -(\tilde{E}_{00}^{(n)})^{-1} \tilde{E}_{01}^{(n)} \tag{6.14}$$

but equation 6.13 is redundant; redundancy corresponds to the fact that equation 6.8 is satisfied identically. The displacement vector obtained in equation 6.14 must satisfy equation 6.13, however, and this condition provides a useful check on the accuracy of the solution. It should be noted that it is not always wise to let first element of the displacement vector be unity; numerical accuracy will be improved if the unit element is associated with one of the larger displacement amplitudes. The same solution process can be employed in any case, however, by merely rearranging the order of the rows and columns of $E^{(n)}$ appropriately.

The displacement amplitudes obtained from equation 6.14 together with the unit amplitude of first component constitute the displacement vector associated with the nth mode of vibration. For convenience the vector is usually expressed in dimensionless form by dividing all the components by one reference component (usually the largest). The resulting vector is called the n^{th} mode shape Φ_n; thus

$$\Phi_n = \begin{bmatrix} \Phi_{1n} \\ \Phi_{2n} \\ \Phi_{3n} \\ \cdot \\ \cdot \\ \cdot \\ \Phi_{Nn} \end{bmatrix} = \frac{1}{u_{kn}} \begin{bmatrix} 1 \\ u_{2n} \\ u_{3n} \\ \cdot \\ \cdot \\ \cdot \\ u_{Nn} \end{bmatrix} \tag{6.15}$$

in which u_{kn} is the reference component.

The shape of each of the N modes of vibration can be found by this same process; the square matrix made up of the N mode shapes will be represented by Φ; thus

$$\Phi = [\Phi_1\ \Phi_2\ \Phi_3 \dots \Phi_N] = \begin{bmatrix} \Phi_{11} & \Phi_{12} \dots\dots\dots\dots \Phi_{1N} \\ \Phi_{21} & \Phi_{22} \dots\dots\dots\dots \Phi_{2N} \\ \Phi_{31} & \Phi_{32} \dots\dots\dots\dots \Phi_{3N} \\ \Phi_{41} & \Phi_{42} \dots\dots\dots\dots \Phi_{4N} \\ \dots\dots & \dots\dots\dots\dots\dots\dots\dots \\ \Phi_{N1} & \Phi_{N2} \dots\dots\dots\dots \Phi_{NN} \end{bmatrix} \tag{6.16}$$

It should be noted that the vibration analysis of a structural system is a form of characteristic-value, or eigenvalue problem of matrix-algebra theory. The frequency-squared term are the eigenvalues and the mode shapes are the eigenvectors.

For 2 degrees of freedom the characteristic equation yields a closed form formula. Consider, for example, a two-story building or coupled response of a rigid body involving one translation and one rotation (Figure 6.3):

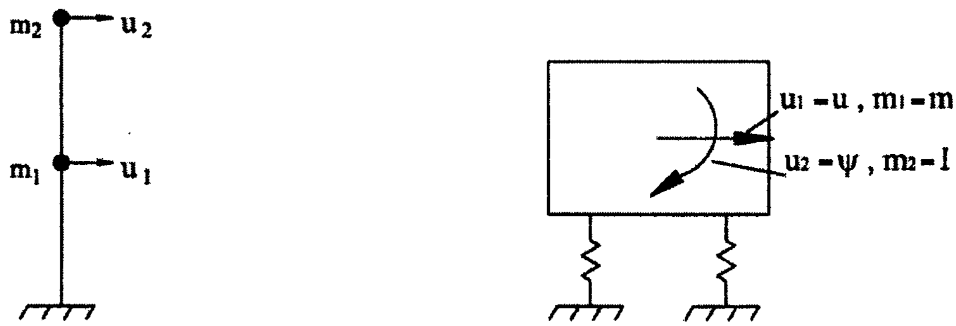


Figure 6.3 System with two DOF.

Then, equation 6.5 becomes

$$\begin{vmatrix} k_{11} - m_1\omega^2 & k_{12} \\ k_{21} & k_{22} - m_2\omega^2 \end{vmatrix} = 0$$

where, $k_{21} = k_{12}$.

The characteristic determinant can be readily calculated:

$$\Delta = \omega^4 m_1 m_2 - \omega^2 (m_1 k_{22} + m_2 k_{11}) + k_{11} k_{22} - k_{12}^2 = 0$$

From here

$$\omega_{1,2}^2 = \frac{1}{2}\left(\frac{k_{11}}{m_1} + \frac{k_{22}}{m_2}\right) \mp \sqrt{\frac{1}{4}\left(\frac{k_{11}}{m_1} - \frac{k_{22}}{m_2}\right)^2 + \frac{k_{12}^2}{m_1 m_2}}$$

with each frequency ω_j, j = 1,2, the amplitude ratios (modes) are from equation 6.4:

$$a_{(j)} = \frac{u_2}{u_1} = \frac{m_1\omega_j^2 - k_{11}}{k_{12}} = \frac{k_{12}}{m_2\omega_j^2 - k_{22}} \qquad , a_{(1)}a_{(2)} = -\frac{m_1}{m_2}$$

Both of these formulas must yield the same answer, which is a check for the correctness of ω_j. The first natural frequency calculated can be checked using an approximate formula based on an energy consideration due Lord Rayleigh.

Rayeligh's formula to check the first natural frequency is

$$\omega_1^2 \cong g\frac{\sum_i Q_i u_i}{\sum_i Q_i u_i^2} = g\frac{\sum_i m_i u_i}{\sum_i m_i u_i^2}$$

where, u_i = static deflection under the dead load of the structure, Q_i acting in the direction of the motion and g = gravity acceleration (Figure 6.4).

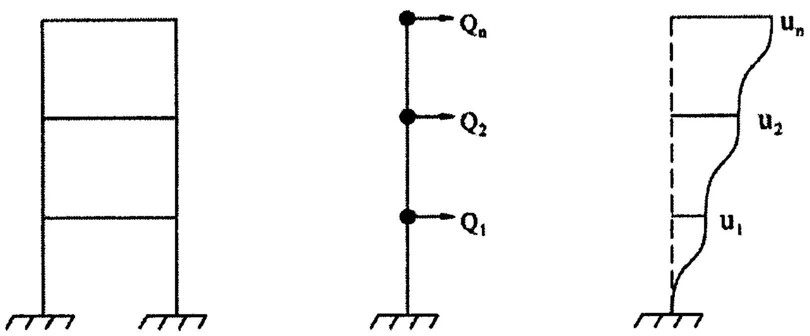


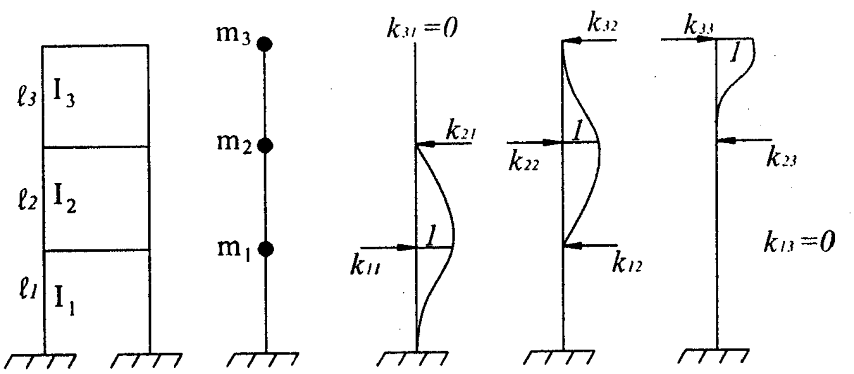
Figure 6.4 Deflection for Rayeligh's formula.

6.2.2 Generation of Stiffness Matrix

Stiffness matrices are particularly easy to generate for shear buildings for which they are assembled using the combination of the basic formula

$$k = 12EI/\ell^3$$

applied to the deflected columns as indicated in Figure 6.5 E.g.:



$$k_{11} = 2\frac{12EI_1}{\lambda_1^3} + 2\frac{12EI_2}{\lambda_2^3}$$

$$k_{21} = 2\times\left(-\frac{12EI_2}{\lambda_2^3}\right)$$

$$k_{31} = 0 \qquad \text{etc.}$$

Figure 6.5 Generation of stiffness constants for shear buildings.

Example: Find the vibration frequencies and mode shapes of the 3-story frame shown in figure 6.6

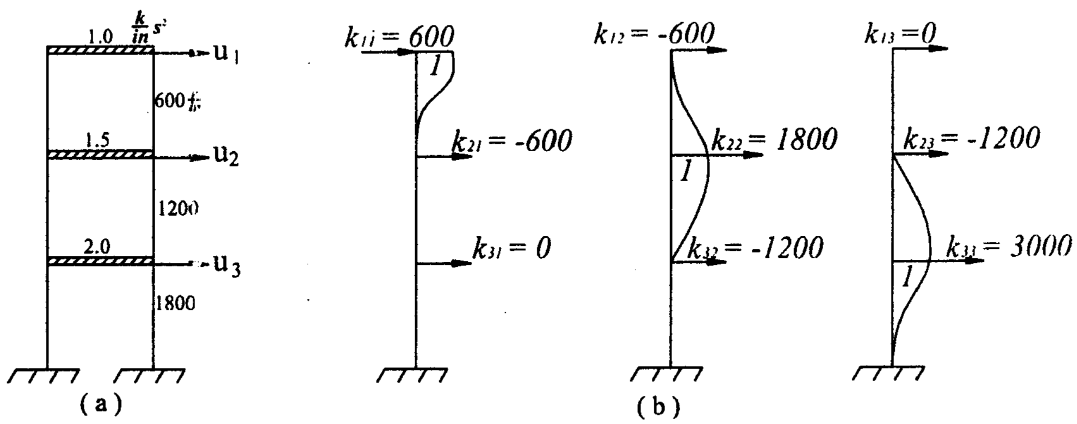


Figure 6.6 Frame used in example of vibration analysis: (a) Structural system; (b) stiffness influence coefficients.

The stiffness matrix for this frame can be determined applying a unit displacement to each story in succession and evaluating the resulting story forces, as shown in the figure. Because the girders are assumed to be rigid, the story forces can easily be determined here by merely adding the sidesway stiffnesses of the appropriate stories.

The mass and stiffness matrices for this frame thus are

$$[m] = (1\,\text{kip.s}^2/\text{in})\begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2.0 \end{bmatrix} \qquad [k] = (600\,\text{kips/in})\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

from which

$$[k]-\omega^2[m] = (600\ \text{kips/in})\left[\begin{array}{c|cc} 1-B & -1 & 0 \\ \hline -1 & 3-1.5B & -2 \\ 0 & -2 & 5-2B \end{array}\right] \tag{a}$$

where, $B \equiv \dfrac{\omega^2}{600}$

The frequencies of the frame are given by the condition that $\Delta = 0$, where Δ is the determinant of the square matrix in equation (a). Evaluating this determinant, simplifying, and equation to zero leads to the cubic equation

$$B^3 - 5.5B^2 + 7.5B - 2 = 0$$

The three roots of the equation may be solved directly or obtained by trial and error; their values are $B_1 = 0.351$, $B_2 = 1.61$, $B_3 = 3.54$. Hence the frequencies are

$$\begin{bmatrix} \omega_1^2 \\ \omega_2^2 \\ \omega_3^2 \end{bmatrix} = \begin{bmatrix} 210 \\ 966 \\ 2{,}124 \end{bmatrix} \qquad \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 14.5 \\ 31.1 \\ 46.1 \end{bmatrix} \text{ rad/s}$$

To obtain the vibration mode shapes use the second and third rows of the vibration matrix (a)

$$\begin{bmatrix} \Phi_{2n} \\ \Phi_{3n} \end{bmatrix} = -\begin{bmatrix} 3-1.5B_n & -2 \\ -2 & 5-2B_n \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \text{(see Eq. 6.14 \& Eq. 6.11)}$$

Thus the mode shapes can be found by introducing the values of B_n, inverting, and multiplying as indicted. The calculations for the three mode shapes of system follow.

Mode 1:

$$B_1 = 0.35 \quad \tilde{E}_{00}^{(1)} = \begin{bmatrix} 2.475 & -2 \\ -2 & 4.300 \end{bmatrix} \quad (\tilde{E}_{00}^{(1)})^{-1} = \frac{1}{6.68}\begin{bmatrix} 4.300 & 2 \\ 2 & 2.475 \end{bmatrix}$$

$$\begin{bmatrix} \Phi_{21} \\ \Phi_{31} \end{bmatrix} = \frac{1}{6.68}\begin{bmatrix} 4.300 \\ 2.00 \end{bmatrix} = \begin{bmatrix} 0.644 \\ 0.300 \end{bmatrix}$$

Mode 2:

$$B_2 = 1.61 \quad \tilde{E}_{00}^{(2)} = \begin{bmatrix} 0.585 & -2 \\ -2 & 1.780 \end{bmatrix} \quad (\tilde{E}_{00}^{(2)})^{-1} = \frac{-1}{2.959}\begin{bmatrix} 1.780 & 2 \\ 2 & 0.585 \end{bmatrix}$$

$$\begin{bmatrix} \Phi_{22} \\ \Phi_{32} \end{bmatrix} = \frac{-1}{2.959}\begin{bmatrix} 1.780 \\ 2.000 \end{bmatrix} = -\begin{bmatrix} 0.601 \\ 0.676 \end{bmatrix}$$

Mode 3:

$$\mathbf{B_3 = 3.54} \quad \tilde{\mathbf{E}}_{00}^{(3)} = \begin{bmatrix} -2.31 & -2 \\ -2 & -2.08 \end{bmatrix} \quad (\tilde{\mathbf{E}}_{00}^{(3)})^{-1} = \frac{1}{0.81}\begin{bmatrix} -2.08 & 2 \\ 2 & -2.31 \end{bmatrix}$$

$$\begin{bmatrix} \Phi_{23} \\ \Phi_{33} \end{bmatrix} = \frac{1}{0.81}\begin{bmatrix} -2.08 \\ 2.00 \end{bmatrix} = \begin{bmatrix} -2.57 \\ 2.47 \end{bmatrix}$$

Of course, the displacement of mass a in each mode has been assumed to be unity. The three mode shapes for this structure are sketched in Figure 6.7

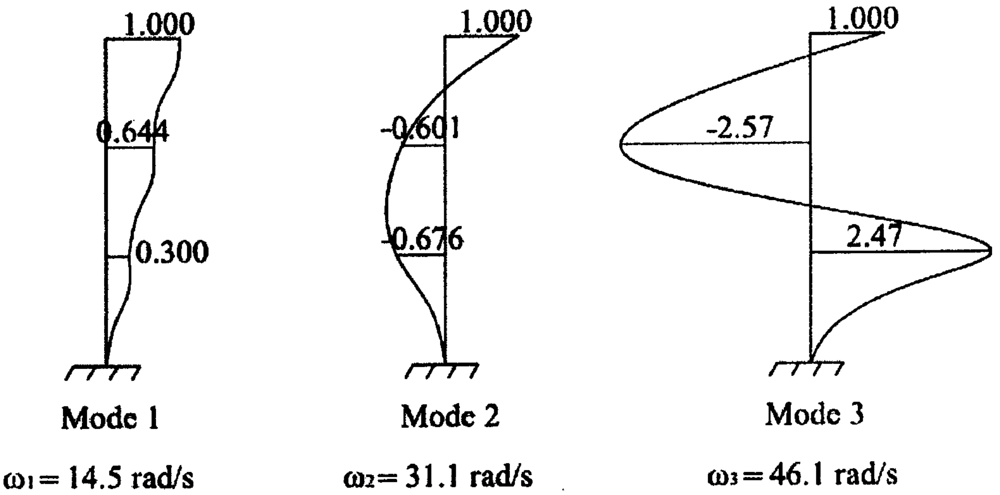
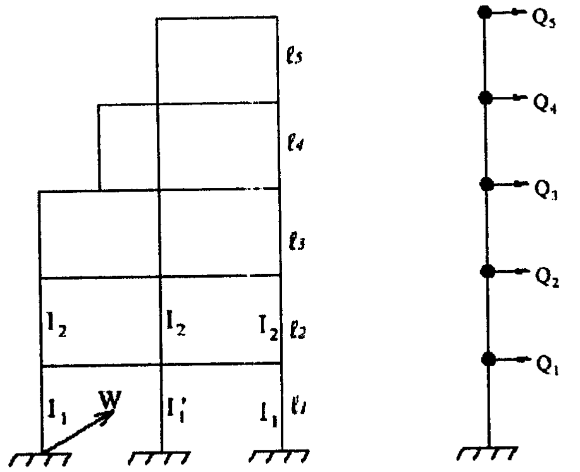


Figure 6.7 Vibration properties for the frame of Figure 6.6.

With more degrees of freedom than 3, it is advisable to organize the whole eigen value problem in terms of matrices and to carry out the calculation using a computer.

Example: A 5 story shear building



ℓ_1 = 13 ft and ℓ_2 to ℓ_5 = 10 ft
$I_1 = 4{,}000\ \text{in}^4$ (depth 24 in)
$I'_1 = 7{,}000\ \text{in}^4$ (depth 26 in)
$I_{2,3} = 3{,}500\ \text{in}^4$ (depth 24 in)
$I_{4,5} = 3{,}000\ \text{in}^4$ (depth 22 in)
$E = 30\times10^6\ \text{psi} = 4.32\times10^9\ \text{lb/ft}^2$
$Q_1 = 70{,}000$ lb
$Q_{2,3} = 60{,}000$ lb
$Q_4 = 45{,}000$ lb
$Q_5 = 30{,}000$ lb

Figure 6.8

<u>Calculate:</u>

a) Stiffness and mass matrices
b) Natural frequencies and modes
c) Participation factor $\mathbf{L_j = \sum_{i=1}^{n} m_i \Phi_{ij}}$ $\qquad$ $\mathbf{j = 1,2...5}$
d) Generalized mass $\mathbf{M_j = \sum_{i=1}^{n} m_i \Phi_{ij}^2}$ $\qquad$ $\mathbf{j = 1,2...5}$

Note: **W** = width of the building needed only in application to wind. Otherwise, only one section (frame) of the building with weights **Q_i** is analyzed.

Mass:

$$\begin{bmatrix} \mathbf{2173.913} & & & & \\ & \mathbf{1863.354} & & \mathbf{0} & \\ & & \mathbf{1863.354} & & \\ & \mathbf{0} & & \mathbf{1397.516} & \\ & & & & \mathbf{931.677} \end{bmatrix} \quad \text{(Slugs)}$$

Stiffness:

$$(\mathbf{10^6})\begin{bmatrix} \mathbf{43.319} & \mathbf{-26.5} & & \mathbf{0} & \\ \mathbf{-26.25} & \mathbf{52.5} & \mathbf{-26.25} & & \\ & \mathbf{-26.25} & \mathbf{48.76} & \mathbf{-22.5} & \\ & \mathbf{0} & \mathbf{-22.5} & \mathbf{37.5} & \mathbf{-15.0} \\ & & & \mathbf{-15.0} & \mathbf{15.0} \end{bmatrix} \quad \text{(lb/ft)}$$

$$\begin{array}{cl} \omega_j := \begin{Bmatrix} 34.715 \\ 94.124 \\ 144.037 \\ 191.553 \\ 222.920 \end{Bmatrix} & \begin{matrix} j=1 \\ =2 \\ =3 \\ =2 \\ =5 \end{matrix} \end{array}$$

$$\begin{array}{cl} \Phi_{ij} := \begin{bmatrix} 0.233 & -0.45 & -0.499 & -0.354 & -0.210 \\ 0.361 & -0.417 & 0.033 & 0.492 & 0.519 \\ 0.461 & -.116 & 0.517 & 0.056 & -0.583 \\ 0.527 & 0.319 & 0.192 & -0.624 & 0.529 \\ 0.57 & 0.709 & -0.666 & 0.488 & -0.253 \end{bmatrix} & \begin{matrix} i=1 \\ =2 \\ =3 \\ =4 \\ =5 \end{matrix} \\ \begin{matrix} j=1 & =2 & =3 & =4 & =5 \end{matrix} & \end{array}$$

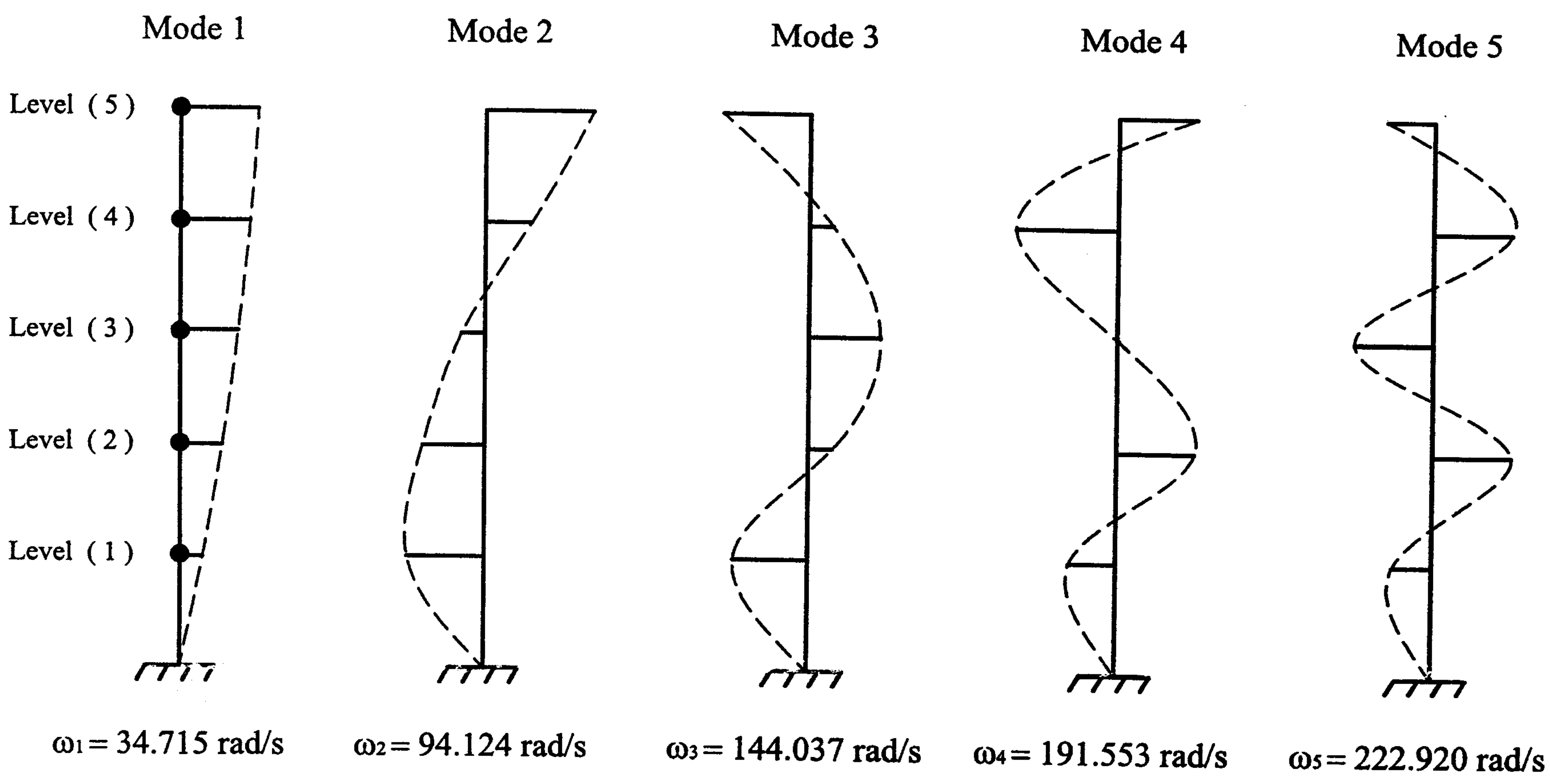
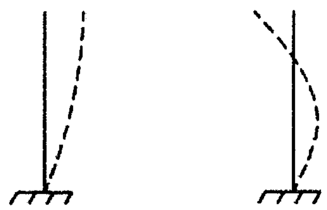


Figure 6.9 **Mode shapes of 5 story building.**

Orthogonality of modes

Orthogonality of modes is a very important relationship between any two modes of free vibration



We found that natural frequencies ω_j and corresponding modes determined by eg. (algebraic):

$$\left([k]-\omega_j^2[`m]\right)\{\Phi_j\}=0, \qquad \{\Phi_j\}=\begin{Bmatrix}\Phi_{1(j)}\\ \Phi_{2(j)}\\ \vdots\\ \Phi_{n(j)}\end{Bmatrix}$$

write this equation for two modes **j** and **k**, e.g. 1-st and 3-rd

$$\omega_j^2[`m]\ \{\Phi_j\}=[k]\{\Phi_j\} \tag{6.18}$$

$$\omega_k^2[`m]\ \{\Phi_k\}=[k]\{\Phi_k\} \tag{6.19}$$

transpose equation 6.18 and postmultiply by $\{\phi_k\}$

$$\left(\omega_j^2[`m]\{\Phi_j\}\right)^T\{\Phi_k\}=\left([k]\{\Phi_j\}\right)^T\{\Phi_k\}$$

Because of the reversal law this is also

$$\omega_j^2\ \{\Phi_j\}^T[`m]^T\{\Phi_k\}=\{\Phi_j\}^T[k]^T\{\Phi_k\} \tag{6.20}$$

(Recall $([\Phi][b]\ \ [c])^T=[c]^T[b]^T[\Phi]^T$).

Matrices $[`m]$ and $[k]$ are symmetric and thus

$[`m]^T=[`m],\ [k]^T=[k]$

Premultiply equation 6.19 by $\{\Phi_j\}^T$

$$\omega_k^2\ \{\Phi_j\}^T[`m]\{\Phi_k\}=\{\Phi_j\}^T[k]\{\Phi_k\} \tag{6.21}$$

The right sides of equations 6.20 and 6.21 are equal and therefore subtracting equation 6.21 from 6.20 yields:

$$\left(\omega_j^2 - \omega_k^2\right) \{\Phi_j\}^T [m] \{\Phi_k\} = 0$$

Since $\omega_j \neq \omega_k$, $\quad \{\Phi_j\}^T [m] \{\Phi_k\} = 0 \qquad$ **for $j \neq k$** (6.22)

This is Orthogonality condition for modes $[\Phi_j]$, $[\Phi_k]$ including mass matrix.

If equation 6.19 is premultiplied by $[\Phi_j]^T$, also by equation 6.22

$$\{\Phi_j\}^T [k] \{\Phi_k\} = 0 \tag{6.23}$$

which is the second orthogonality condition including stiffness matrix.

Equation 6.22 is

$$\underset{1\times n}{\{\Phi_1 \Phi_2 \ldots\ldots\ldots\ldots \phi_n\}_j} \underset{n\times n}{\begin{bmatrix} m_1 & 0 & \ldots & 0 \\ 0 & m_2 & \ldots & . \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ 0 & . & \ldots & m_n \end{bmatrix}} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ . \\ . \\ . \\ \Phi_n \end{Bmatrix}_k = 0$$

$$\underset{1\times n}{\{m_1\Phi_{1j} \;\; m_2\Phi_{2j} \ldots\ldots\ldots\ldots m_n\Phi_{nj}\}} \underset{n\times 1}{\begin{Bmatrix} \Phi_{1k} \\ \Phi_{2k} \\ . \\ . \\ . \\ \Phi_{nk} \end{Bmatrix}} = 0$$

$$m_1\Phi_{1j}\,\Phi_{1k} + m_2\Phi_{2j}\Phi_{2k} + \ldots\ldots\ldots\ldots m_n\Phi_{nj}\Phi_{nk} = 0$$

or finally,

$$\sum_{i=1}^{n} m_i\Phi_{ij}\,\Phi_{ik} = 0 \qquad \text{for } j \neq k$$

GENERALIZATION OF ORTHOGONALITY CONDITIONS

It was found that between two different modes **for $j \neq k$, $\omega_j \neq \omega_k$** and the orthogonality conditions is

$$\{\Phi_j\}^T [m] \{\Phi_k\} = 0 \quad \text{or} \sum_i m_i\Phi_{ij}\Phi_{ik} = 0 \tag{6.24}$$

for $j = k$ $\sum_i m_i \Phi_{ij}^2 \neq 0$ because $m_i > 0$ and $\Phi_{ij}^2 > 0$. Thus

$$\{\Phi_j\}^T [\grave{}m] \{\Phi_j\} = \sum_i m_i \Phi_{ij}^2 = M_j$$

where, M_j = generalized mass of the j-Th mode.
Therefore, more generally for all the modes

$$[\Phi]^T [\grave{}m][\Phi] = [\grave{}M] = \text{diagonal matrix of generalized masses} \tag{6.25}$$

This can be checked by rewriting $[\Phi]$ in terms of partitioned matrices and treating the submatrices as elements if they are conformable. Thus, triple matrix product

$$[\Phi]^T [\grave{}m][\Phi]$$

is also

$$\begin{array}{r} \text{Ist mode} \\ \\ \text{2nd} \rightarrow \\ \\ \\ \\ \text{n-th} \rightarrow \end{array} \begin{Bmatrix} \{\Phi_1\}^T \\ ---- \\ \{\Phi_2\}^T \\ \cdot \\ \cdot \\ \cdot \\ \{\Phi_n\}^T \end{Bmatrix} [m][\{\Phi_1\} \vdots \{\Phi_2\} \cdots \quad \vdots \{\Phi_n\}]$$

$$n \times 1 \qquad n \times n \qquad 1 \times n \qquad \text{- is conformable, } n \times n$$

$$= \begin{Bmatrix} \{\Phi_1\}^T [m] \\ \{\Phi_2\}^T [m] \\ ----- \\ \vdots \\ \{\Phi_n\}^T [m] \end{Bmatrix} [\{\Phi_1\} \vdots \{\Phi_2\} \cdots \quad \vdots \{\Phi_n\}] = \begin{bmatrix} M_1 & 0..... & 0 \\ 0 & M_2 & 0 \\ 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & M_n \end{bmatrix}$$

The second orthogonality condition involves the stiffness

$$\{\Phi_j\}^T [k] \{\Phi_k\} = 0, \qquad j \neq k$$

$$\{\Phi_j\}^T [k] \{\Phi_j\} = K_j, \qquad j = k \tag{6.26}$$

in which, K_j = generalized stiffness (1 x 1 matrix = scalar).

From equation 6.26, a relation can be derived involving all the modes, written as columns in $[\Phi]$:

$$[\Phi]^T[k][\Phi] = [`K] = \omega_j^2\,[`M] \tag{6.27}$$

in which, with regard to equation 6.21,
$[K]$ = generalized stiffness matrix; ω_j = j-th natural frequency.

To prove equation 6.27, $[\Phi]^T$ and $[\Phi]$ can be partitioned according to the modes and the matrices multiplied, i.e.

$$[\Phi]^T[k]\ [\Phi] =$$

$$\underset{n\times 1}{\begin{Bmatrix} \{\Phi_1\}^T \\ ---- \\ \{\Phi_2\}^T \\ ---- \\ \cdot \\ \cdot \\ \cdot \\ \{\Phi_n\}^T \end{Bmatrix}} \underset{n\times n}{[k]} \underset{1\times n}{[\{\Phi_1\}:\{\Phi_2\}:\ldots\ldots\ldots\ldots:\{\Phi_n\}]}$$

$$\underset{n\times 1}{\begin{Bmatrix} \{\Phi_1\}^T[k] \\ ----- \\ \{\Phi_2\}^T[k] \\ ----- \\ \cdot \\ \cdot \\ \cdot \\ \{\Phi_n\}^T[k] \end{Bmatrix}} \underset{1\times n}{[\{\Phi_1\}:\{\Phi_2\}:\ldots\ldots\ldots\ldots:\{\Phi_n\}]} = \begin{bmatrix} K_1 & 0\ldots & 0 \\ 0 & K_2 & 0 \\ \vdots & \vdots & \\ 0 & 0\ldots & K_n \end{bmatrix} = [K] = [\omega_j^2][M]$$

With respect to equation 6.20 written for $j = k$

$$K_j = \{\Phi_j\}^T[k][\Phi_j] = \omega_j^2\,[\Phi_j]^T[`m]\{\Phi_j\} = \omega_j^2 M_j$$

6.3 FORCED VIBRATION OF MDOF SYSTEM BY MODAL ANALYSIS

Modal analysis is a general method for analyzing the response of linear multi-degree-of-freedom system. It is particularly suitable for systems whose properties are frequency independent. The method describes the response in terms of the modes of free vibration whose orthogonality facilitates the solution. Therefore, the analysis of the free vibration (the solution of the eigenvalue problem) must be completed prior to the calculation of the response to external excitation.

The equations of motion can be written in the general form as

$$[m]\{\ddot{u}\}+[c]\{\dot{u}\}+[k]\{u\}=\{p\} \tag{6.28}$$

in which **[m]**, **[c]** and **[k]** are mass, damping, and stiffness matrices respectively; **{u}** = the displacement vector and **{p}** = the vector of excitation.

The solution to equation 6.28 describing the response is sought in the form of a sum of responses in individual modes of free vibration, Φ_{ij},

$$u_i(t)=\sum_{j=1}^{n}\Phi_{ij}\eta_j(t), \qquad i=1,2...n \tag{6.29}$$

in which,

Φ_{ij} are mode coordinates of the **j**-th mode. These mode coordinates are independent of time and can be chosen to arbitrary scale. [In systems with distributed mass, $u_i(t)$ is replaced by $u(x,t)$ and Φ_{ij} by $\Phi_j(x)$];

$\eta_j(t)$ are new variables associated with mode **j** and depending on time. They are called ***generalized coordinates***. Equation 6.29 can be interpreted as Figure 6.10 indicates.

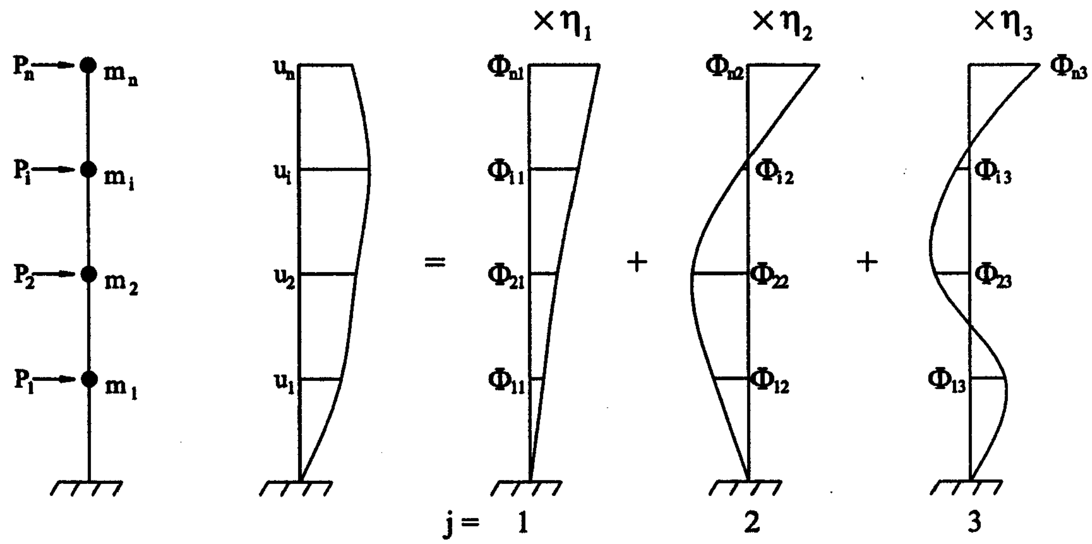


Figure 6.10 Response in terms of modes.

Equation 6.29 represents a ***coordinate transformation*** through which one set of n coordinates can be replaced by another set of n independent coordinates. Equation 6.29 can be rewritten in matrix form to include all nodes, i = 1,2..., n,

$$[u] = [\Phi]\{\eta\}, \qquad [\dot{u}] = [\Phi]\{\dot{\eta}\}, \qquad [\ddot{u}] = [\Phi]\{\ddot{\eta}\}, \tag{6.30}$$

where,

$$[\Phi] = \begin{bmatrix} \Phi_{11} & \vdots & \Phi_{12} & & \Phi_{1n} \\ \Phi_{21} & \vdots & \Phi_{22} & & \Phi_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ \Phi_{n1} & \vdots & \Phi_{n2} & & \Phi_{nn} \end{bmatrix} \tag{6.31}$$

Each column in equation 6.31 represents one mode of free vibrations.

Substitute equations 6.30 into equation 6.28 and premultiply by the transpose of $[\Phi]$ which is $[\Phi]^T$; (this is a matrix in which modes are presented in rows).

$$\underbrace{[\Phi]^T[`m][\Phi]}_{[`M]}\{\ddot{\eta}\} + [\Phi]^T[c][\Phi]\{\dot{\eta}\} + \underbrace{[\Phi]^T[k][\Phi]\{\eta\}}_{[`K]} = [\Phi]^T\{p(t)\} \tag{6.32}$$

Equation 6.32 considerably simplifies due to the generalized orthogonality conditions, discussed previously, according to which

$$[\Phi]^T[`m][\Phi] = [`M] \tag{6.33a}$$

$$[\Phi]^T[k][\Phi] = [`K] = [\omega_j^2][`M] \tag{6.33b}$$

Hence, the two triple products result in two diagonal matrices which is very advantageous because $[M]\{\ddot{\eta}\}$ and $[K]\{\eta\}$ are column matrices. Therefore, each equation 6.32, written in ordinary algebraic form, contains only one variable η_j and its second derivation $\ddot{\eta}_j$.

Clearly, it would be most desirable if the triple matrix product

$$[\Phi]^T[c][\Phi] \tag{6.34}$$

containing the damping constants of the system, resulted in a diagonal matrix because only then may each equation (line) contain only one derivative $\dot{\eta}_j$. In such a case equation 6.32 represents a set of **n** independent equations for η_j, j = 1,2..., n, that are "uncoupled".

Since $[\Phi]^T$ and $[\Phi]$, two multipliers, are the same in equation 6.34 and equation 6.33, the triple matrix product, equation 6.34 can result in a diagonal matrix only when the damping matrix **[c]**, is proportional to either the mass matrix $[`m]$ or the stiffness matrix **[k]**. In the first case, [c] has to be diagonal and proportional to **[m]**,

$$[c] = 2\alpha[\ulcorner m\,] \tag{6.35a}$$

while the second case occurs if

$$[c] = \beta[k] \tag{6.35b}$$

Factor 2 is used for convenience, α, β = constants. (Recall that in one degree of freedom the viscous damping constant $c = 2\alpha m$). Equation 6.35a substituted into equation 6.34, yields with respect to equation 6.33 a

$$[\Phi]^T[c][\Phi] = 2\alpha[\ulcorner M\,] \tag{6.36a}$$

and equation 6.35b gives

$$[\Phi]^T[c][\Phi] = \beta[\ulcorner \omega_j^2\,][\ulcorner M\,] \tag{6.36b}$$

Equation 6.35a implies that only absolute dampers may be present, every damping constant, c_i is proportional to the mass m and, finally, the proportionality constant α is the same for all dampers.

Equations 6.33a and 6.36a substituted into equation 6.32 give

$$[\ulcorner M\,]\{\ddot{\eta}\} + 2\alpha[\ulcorner M\,]\{\dot{\eta}\} + [\ulcorner \omega_j^2\,][\ulcorner M\,]\{\eta\} = [\Phi]^T\{P(t)\} \tag{6.37}$$

This is set of n independent (uncoupled) equations. Each of them has the form

$$M_j\ddot{\eta}_j + 2\alpha M_j\dot{\eta} + \omega_j^2 M_j\eta_j = [\Phi_j]^T\{P(t)\} \tag{6.38}$$

or

$$\ddot{\eta}_j(t) + 2\alpha\dot{\eta}_j(t) + \omega_j^2\eta_j(t) = \frac{P_j(t)}{M_j} \qquad ,j = 1,2,\ldots,n \tag{6.39}$$

in which

$$P_j(t) = \{\Phi_j\}^T\{P(t)\} = \sum_{i=1}^{n} \Phi_{ij}\ P_i(t) \tag{6.40}$$

is the generalized force linked to generalized coordinate η_j. Equations 6.39 are independent and each of them is exactly equal to the equation of a single degree of freedom system and, therefore, can easily be solved. Thus, whenever one degree of freedom can be solved, many degrees of freedom can be solved too; η_j are obtained from equations 6.39 and substituted into equation 6.29.

The uncoupled generalized coordinates $\eta_j(t)$ are also called ***normal coordinates***.

The damping constant α occurring in equation 6.39 is calculated for each mode as $\alpha = D_j\ \omega_j$ in analogy with one degree of freedom. The modal damping ratio, D_j, can be in some cases calculated, e.g. damping due to soil, fluids or air, in other cases is estimated.

With damping matrix proportional to stiffness matrix (equation 6.35b), uncoupled equation 6.39 is again obtained with 2 α replaced by $\beta\omega_j^2$.

The decoupling of the equations of motion can be interpreted as a result of the external loads being in effect replaced by a system of fictitious (generalized) loads such that each of them excites just one mode only.

Subscript **i** refers to a component of motion or to mass **i**, subscript j identifies the mode. The approach is general with number of masses $\rightarrow \infty$ a discrete system changes into a distributed one. The only change in modal analysis is that $n = \infty$ and summations in the equations for generalized, mass and generalized force change to integration:

$$M_j = \int_0^{\ell} m(x)\Phi_j^2(x)dx$$

$$P_j = \int_0^{\ell} p(x,t)\Phi_j(x)dx$$

Equation 6.39 and 6.29 remain unchanged.

Further solution depends only on the character of the external forces. The principal types of excitation are discussed below.

If the damping is not proportional to either [k] or [m], the equations of motion can be uncoupled using complex vibration modes.

6.3.1 Harmonic Excitation

Assume harmonic excitation with frequency Ω as in the case of unbalanced masses of machines, vortex shedding etc. Such forces can be described as

$$P_i(t) = P_i \cos\Omega t \quad \text{or} \quad \{p(t)\} = \{P\}\cos\Omega t \tag{6.41}$$

where.

$$\{P\} = [P_1 P_2 ... P_i ....P_n]^T$$

The generalized forces are from equation 6.40

$$P_j(t) = \cos\Omega t \sum_{i=1}^{n} \Phi_{ij}P_i \tag{6.42}$$

or

$$P_j(t) = L_j \cos\Omega t, \quad L_j = \sum_{i=1}^{n} \Phi_{ij}P_i \tag{6.43}$$

The generalized equation of motion, equation 6.39

$$\ddot{\eta}_j(t) + 2\alpha\dot{\eta}_j(t) + \omega_j^2\eta(t) = \frac{L_j}{M_j}\cos\Omega t \tag{6.44}$$

This is an equation formally identical with that of single degree of freedom systems whose mass is M_j and which is subjected to harmonic excitation with amplitude L_j. This L_j is called the force participation factor because its magnitude is a measure of the degree to which the excitation forces participate in the excitation of mode **j**.

The solution of equation 6.44 follows from equation 4.15 found previously,

$$\eta_j(t) = \eta_j\cos(\Omega t + \theta_j) + \eta_j^0 e^{-\alpha t}\cos(\omega_j t + \theta_j^0) \tag{6.45}$$

in which the first term describes the most important steady state part of the motion (particular solution). Its amplitude is

$$\eta_j = \frac{L_j}{M_j\omega_j^2}\frac{1}{\sqrt{[1-(\frac{\Omega}{\omega_j})^2]^2 + 4(\frac{\Omega}{\omega_j})^2 D_j^2}} = \frac{L_j}{M_j\omega_j^2}\varepsilon_j \tag{6.46}$$

where, ε_j the dynamic magnification factor in one degree-of freedom whose natural frequency is ω_j damping D_j and M_j ω_j^2 $=K_j$ generalized stiffness. The phase shift of the steady state component

$$\theta_j = -\arctan\frac{2D_j\frac{\Omega}{\omega_j}}{1-(\frac{\Omega}{\omega_j})^2} \tag{6.47}$$

The transient part of equation 6.45 dies out due to damping; constants η_j^0 and θ_j^0 if needed, are given by initial conditions.

The real steady motion is from equation 6.29

$$u_i(t) = \sum_{j=1}^{n}\Phi_{ij}\ \eta_j = \sum_{j=1}^{n}u_{ij}\cos(\Omega t + \theta_j) \tag{6.48}$$

where, the amplitude in mode **j** is

$$u_{ij} = \frac{L_j\Phi_{ij}}{M_j\omega_j^2}\frac{1}{\sqrt{[1-(\frac{\Omega}{\omega_j})^2]^2 + 4(\frac{\Omega}{\omega_j})^2 D_j^2}} = \frac{L_j\Phi_{ij}}{M_j\omega_j^2}\varepsilon_j \tag{6.48a}$$

Thus, the response in each displacement coordinate **i** consists of harmonic components that have the same frequency **Ω** but different amplitudes and phase shifts. At resonance with mode **r**, $\Omega = \omega_j = \omega_r$, $\varepsilon_j = 1/2D_r$; the resonant amplitude of the resonating mode and its phase are

$$u_{ir} = \frac{L_r \Phi_{ir}}{M_r \omega_r^2} \frac{1}{2D_r}, \quad \theta_r = \pi/2 \tag{6.49}$$

With small damping, the resonant amplitude is usually much larger than the nonresonant amplitudes of the other modes and equation 6.49 is sufficient to estimate the resonant amplitudes of the system.

The superposition of the responses in individual modes is shown in figure 6.11 because of the phase differences of the modal components, the resultant amplitude:

$$u_i \le |u_{i1}| + |u_{i2}| + |u_{i3}| + \ldots\ldots\ldots\ldots$$

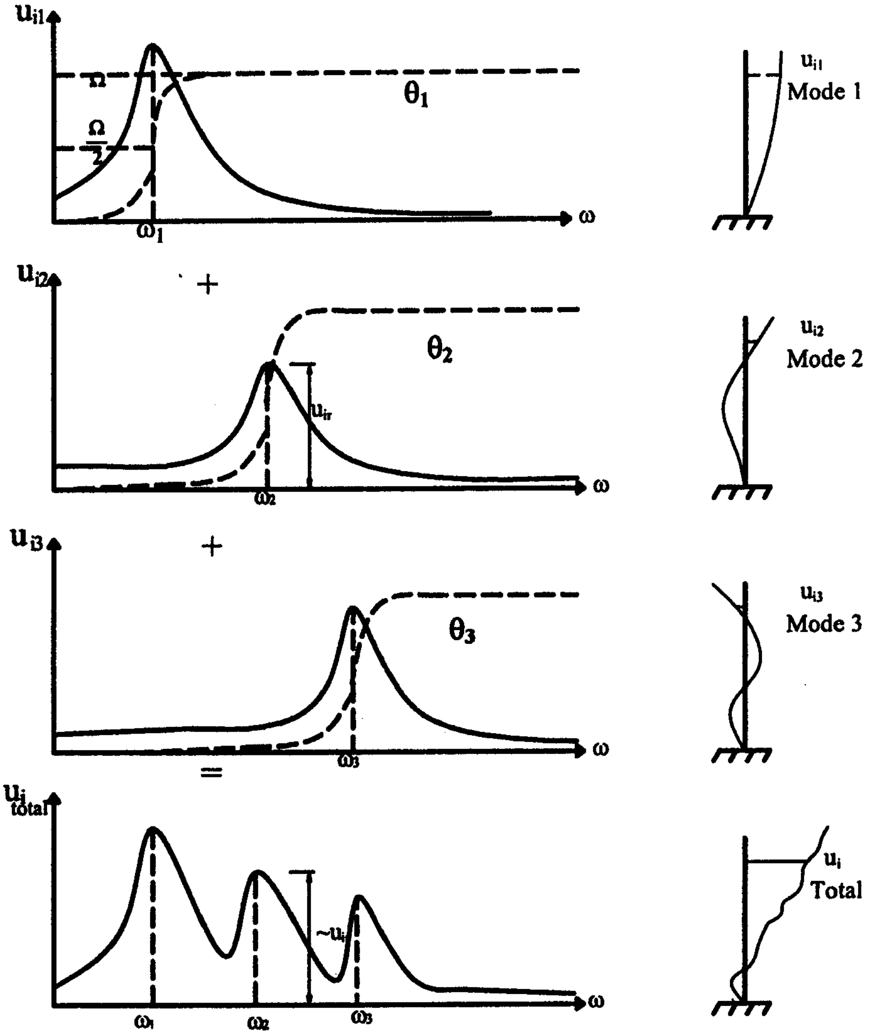


Figure 6.11 Superposition of responses in individual modes.

When using the modal analysis, the coordinates of the modes (eigen-vectors) can be chosen to an arbitrary scale. This can be seen from equation 6.48 and from the application of this formula to one degree of freedom.

In ***one degree*** of freedom,

$$\omega_j^2 = k/m, \quad L_r = \Phi p, \quad M_r = m\Phi^2, \quad \Phi_{ir} = \Phi$$

and the resonant amplitude from equation 6.49.

$$u = \frac{\Phi p \Phi m}{m\Phi^2 k 2D} = \frac{p}{k}\frac{1}{2D} = u_{st}\frac{1}{2D}$$

as found previously.

Before the model analysis is started, the modes are sometimes normalized in such a way that $\Phi_{nj} = 1 \text{ or } M_j = 1$ for each mode. In the latter case, the modes so normalized are called orthogonal modes. Their coordinates are

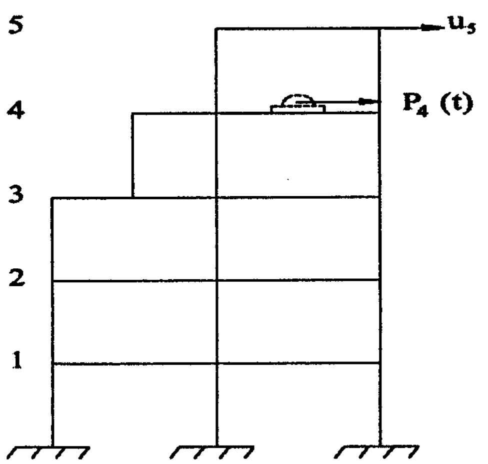
$$\overline{\Phi}_{ij} = \frac{\Phi_{ij}}{\sqrt{M_j}}$$

and the generalized mass

$$\overline{M}_j = \sum_{i=1}^{n} m_i \overline{\Phi}_{ij}^2 = 1$$

This normalization is used by some writers but it does not offer any particular advantage.

<u>Problem:</u> Consider the five story shear building.



The elevator drive produces a harmonic fore acting on the 4th floor, $P_4(t)=P_4 \cos \Omega t$
P_4 = 300 lbs $\simeq$ 1.5 kN. Frequency Ω is variable.

Find the resonant amplitudes of the top (5th) floor at natural frequencies $\Omega = \omega_1$, $\Omega = \omega_2$ $\Omega = \omega_3$ and also amplitudes at the operating speed $\Omega = 1/2\,(\omega_3 + \omega_4)$.

Damping ratio for all modes is $D_j = 0.01$ and thus, $\alpha_j = D\omega_j$

6.3.2 Static Loading

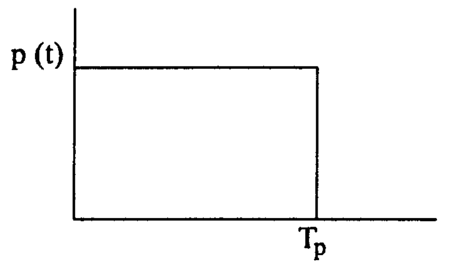
Static loading can also be conveniently solved by means of modal analysis as a special case of harmonic excitation. The pertinent formulae follow from the preceding paragraph with the frequency of excitation $\Omega \to 0$ and thus $\cos \Omega t = 1$. With $\Omega = 0$, loads $P_i(t) = P_i$ and $\theta = 0$. From equation 6.46 the generalized coordinate of mode **j** is

$$\eta_{j,st} = \frac{L_j}{M_j \omega_j^2} \tag{6.50}$$

and from equation 6.48 the static displacement of mass m_i due to static loads P_i ($i = 1,2,\ldots,n$),

$$u_{i,st} = \sum_{j=1}^{n} \frac{L_j}{M_j \omega_j^2} \Phi_{ij} \tag{6.51}$$

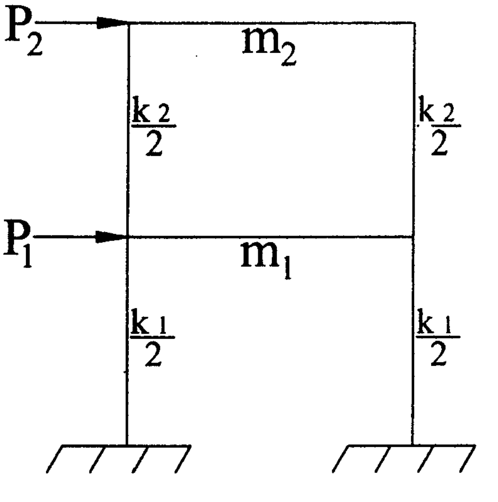
This is an exact approach, suitable to examine the effect of static loads in statically indeterminate structures if the free vibration modes are known already from previous analysis.
This approach can also be used to find the response to a suddenly applied static load or a rectangular pulse whose duration T_p is sufficiently longer than the period of the structure T_1.In these cases, the maximum (peak) response $\leq 2 \times$ static response.



Problem (1):

The two-story shear building is exposed to static wind load in the horizontal direction given as $P_1 = 30$ kN, $P_2 = 20$ kN.

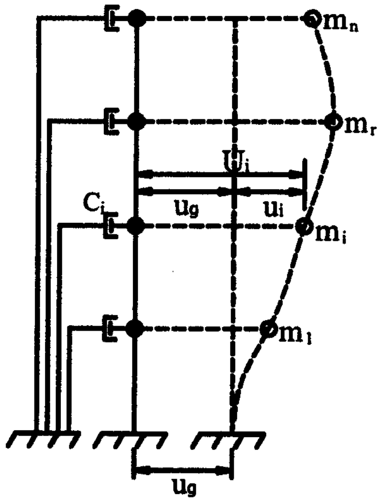
Compute the displacement u_1, u_2 and the stresses in the columns.



6.3 Response to Ground Motion

Differential equations of motion depend on the nature of damping. The more important case is that of relative damping.

Relative damping – From Newton's second law, forces acting on mass $\mathbf{m_i}$ are



$$m_i\ddot{U}_i = -\sum_{r=1}^{n} k_{ir}(U_i - u_g) - c_i(\dot{U}_i - \dot{u}_g)$$

where the absolute displacement $U_i = u_g + u_i$
and the relative displacement $u_i = U_i - u_g$
Whit the damping force at mass $\mathbf{m_i}$ described as

$$-c_i(\dot{U}_i - \dot{u}_g) = -2m_i\alpha(\dot{U}_i - \dot{u}_g)$$

the differential equations of the motion in terms of relative coordinates **u** become

$$m_i\ddot{u}_i + m_i\alpha\dot{u}_i + \sum_{r=1}^{n} k_{ir}u_r = -m_i\ddot{u}_g, \qquad i = 1,2,\ldots n \tag{6.52}$$

This is formally equal to equation 2.18 in which the equivalent exciting forces

$$P_i(t) = -m_i\ddot{u}_g, \qquad i = 1,2,\ldots n$$

The damping satisfies equation 6.35a and, therefore, modal analysis leads to decoupled equations for generalized coordinates $\mathbf{n_j}$. Generalized force (the sign can be omitted) is

$$P_j(t) = \sum_i P_i(t)\Phi_{ij}, \quad \text{ie.}$$

$$P_j(t) = \ddot{u}_g\sum_{i=1}^{n} m_i\Phi_{ij} = \ddot{u}_g L_j$$

where, the earthquake participation factor $L_j = \sum_{i=1}^{n} m_i\Phi_{ij}$.

In matrix form, equations 6.52 can be written as

$$[m]\{\ddot{u}\} + [c]\{\dot{u}\} + k\{u\} = -[m]\{1\}\ddot{u}_g \tag{6.52a}$$

in which, $[c] = 2\,\alpha[m]$ and $\{1\} = [1\ 1\ \ldots\ 1]^T$.

Equation 6.52a can also be derived directly realizing that the inertia forces stem from the absolute displacements U_i, but the stiffness and damping forces are due to the relative displacement u_i. Then,

$$[m]\{\ddot{U}\} + [c]\{\dot{u}\} + [k]\{u\} = 0$$

Eliminating U, equation 6.52a is obtained.

Relative displacement of mass i is, by equation 6.29

$$u_i(t) = \sum_{j=1}^{n} \Phi_{ij}\eta_j \tag{6.52b}$$

in which the generalized coordinate is given by equation 6.39

$$\ddot{\eta}_j + 2\alpha\dot{\eta}_j + \omega_j^2\eta_j = \frac{P_j(t)}{M_j} = \frac{L_j}{M_j}\ddot{u}_g(t) \tag{6.53}$$

This is analogous to one degree of freedom in which the solution depends on the type of ground motion.

Transient ground motion produces response whose solution in 1 DOF is given by the Duhamel (convolution) integral

$$y(t) = \frac{1}{m\omega_0}\int_0^t P(\tau)e^{-D\omega_0(t-\tau)}\sin\omega_0(t-\tau)d\tau$$

with $P(\tau) = L_j\ddot{u}_g(\tau)$, substitution gives for the generalized coordinate

$$\begin{aligned}\eta_j(t) &= \frac{1}{\omega_j}\frac{L_j}{M_j}\int_0^t \ddot{u}_g(\tau)e^{-D_j\omega_j(t-\tau)}\sin\omega_j(t-\tau)d\tau \\ &= \frac{1}{\omega_j}\frac{L_j}{M_j}V_j(t)\end{aligned}$$

where,

$$V_j(t) = \int_0^t \ddot{u}_g(\tau)e^{-D_j\omega_j(t-\tau)}\sin\omega_j(t-\tau)d\tau \tag{6.54}$$

The complete response is the sum of responses in individual modes, i.e.,

$$u_j(t) = \sum_{j=1}^{n} \frac{1}{\omega_j} \frac{L_j}{M_j} \Phi_{ij} V_j(t) \tag{6.55}$$

Integral $V_j(t) = f(\ddot{u}_g, \omega_j, D_j)$ is exactly the same as in one degree of freedom and can be obtained by numerical integration. **V(t)** has the dimension of velocity, e.g. in/s or m/s. Equation 6.55 can be interpreted as Figure 6.12 indicates.

The ***spectral Approach*** to Transient Motion

It is usually not necessary to find the complete time history of the response. The maximum response is decisive in most applications and this is obtained by substituting the maximum value of integral $V_j(t)$ into equation 6.55 using the same notation as in one degree of freedom

$V_{max} = S_v$ = spectral velocity (pseudo velocity)

$S_d \quad = \frac{V_{max}}{\omega_j} = \frac{S_v}{\omega_j} = \frac{S_a}{\omega_j^2}$ = spectral displacement

$S_a \quad = \omega_j S_v$ = spectral acceleration.

Spectral velocities can be computed for any typical earthquake. In Figure 5.3, spectral velocity of El-Centro earthquake is given. Smoothed or averaged spectra should be used possibly depending on site conditions. In terms of spectral displacement, the maximum (peak) displacement in the j-th mode is

$$\hat{u}_{ij} = \frac{1}{\omega_j} \frac{L_j}{M_j} \Phi_{ij} S_v(j) = \frac{L_j}{M_j} \Phi_{ij} S_d(j) \tag{6.56}$$

The effective acceleration for mode j is

$$\ddot{u}_{ij} = \omega_j^2 \hat{u}_{ij} \tag{6.57}$$

the stresses can be computed with a static load equal to the maximum effective earthquake force acting on m_i in mode j defined as

$$q_{ij} = m_i \times (\text{accel.amplitude}) = m_i \omega_j^2 \hat{u}_{ij} = m_i \frac{L_j}{M_j} \Phi_{ij} S_a(j) \tag{6.58}$$

The total maximum base shear, which is a measure of earthquake loading, is

$$Q_j = \sum_{i}^{n} q_{ij} = \frac{L_j}{M_j} S_a(j) \sum_{i} m_i \Phi_{ij} = \frac{L_j^2}{M_j} S_a(j) \tag{6.59}$$

The total response is a sum of responses in individual vibration modes (Figure 6.12).

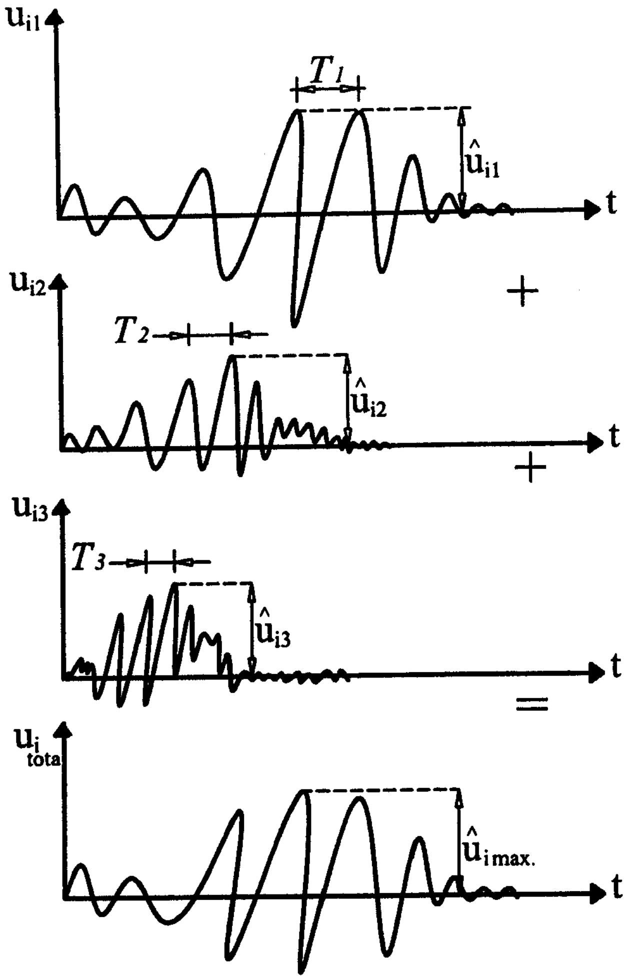


Figure 6.12

The peak in individual modes does not appear at the same time. Its accurate value could be obtained from the resultant time history. Approximately, the maximum response of mass $\mathbf{m_i}$ is

$$\hat{u}_{i,max} \cong \sqrt{\hat{u}_{i,1}^2 + \hat{u}_{i,2}^2 + \ldots} = \sqrt{\sum_j \hat{u}_{i,j}^2}$$

with ***harmonic motion*** of the ground

$$u_g(t) = u_g \cos\Omega t, \qquad \ddot{u}_g(t) = -u_g\Omega^2 \cos\Omega t$$

and loads

$$P_i(t) = m_i\ddot{u}_g(t) = m_i\Omega^2 u_g \cos\Omega t$$

Steady state response is from (6.46) and (6.48)

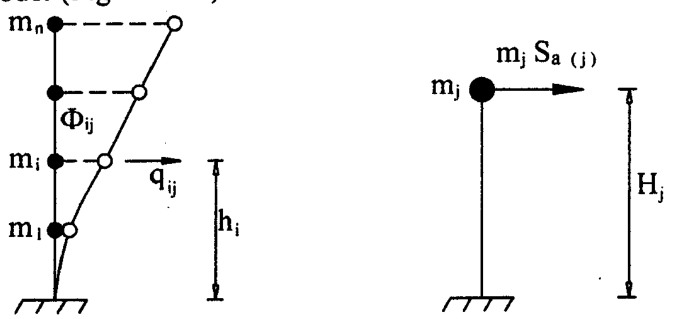
$$u_i(t) = u_g \sum_j \frac{L_j\,\Omega^2}{L_j\,\omega_j^2} \frac{1}{\sqrt{\left[1-\dfrac{\Omega^2}{\omega_j^2}\right]^2 + 4\dfrac{\Omega^2}{\omega_j^2}D_j^2}} \Phi_{ij}\cos(\Omega t + \theta_j)$$

The resonant amplitude in mode j is

$$u_i(r) = u_g \frac{L_j}{M_j}\Phi_{ij}\frac{1}{2D_j}$$

6.4 MODAL EQUIVALENT MODEL

In earthquake engineering the "modal equivalent model" is sometimes used to represent the structure. This model comprises n single degree of freedom system whose damping and natural frequencies are identical to those of individual vibration modes and the masses and their heights are so determined that identical base moments and base shears are obtained from both mode j and the equivalent model (Figure 6.13).



Mode j

Equivalent 1 DOF Model

Figure 6.13 Structural response in mode j and its equivalent 1 DOF model.

Thus, for the base shear:

$$Q_j = \frac{L_j^2}{M_j} S_a(j) = m_j S_a(j) \tag{6.60}$$

and from here the equivalent mass is

$$m_j = \frac{L_j^2}{M_j}. \tag{6.61}$$

The equality of base moments (overturning moments) requires

$$\sum_i q_{ij} h_i = m_j S_a(j) H_j \tag{6.62}$$

which yields

$$H_j = \frac{\sum_i q_{ij} h_i}{L_j^2 S_a(j)} M_j = \frac{\sum_i m_i \frac{L_j}{M_j} \Phi_{ij} S_a(j) h_i}{L_j^2 S_a(j)} M_j \tag{6.63}$$

After abbreviation, the equivalent height is

$$H_j = \frac{\sum_i m_i \Phi_{ij} h_i}{L_j}$$

All the modes can be represented as Figure 6.14 indicates.

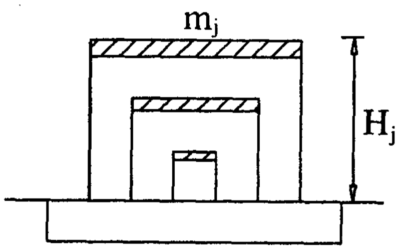


Figure 6.14 Representation of an n-degrees of freedom structure by n 1 DOF system.

Problems

1. Calculate the earthquake response, i.e. displacements, earthquake forces, base shear and stresses in the lowest columns of the five story shear building due to the 20% El-Centro (Figure 5.3). Consider all modes and damping ratio 2%.

2. Calculate the resonant amplitudes of the five story shear building due to harmonic horizontal ground motion $\mathbf{u_g}$ **(t)** whose amplitude is 0.01 in and frequency ranges from 0 to 1.2 ω_5. Assume $\mathbf{D_j}$ = 0.01.

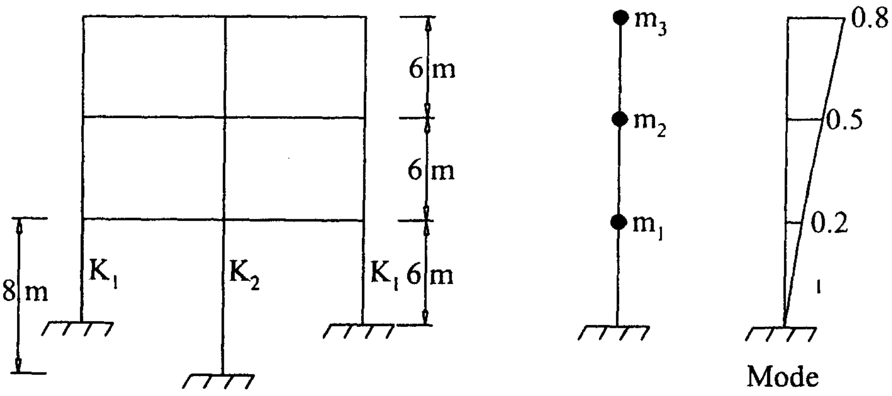
Answers to Problem 1

amplitudes : u_{51} = 0.261 in f_1 = 5.66 ω_1 = 35
: u_{52} = 0.00564 in f_2 = 15.16
: u_{53} = 0.0006 in

Base shear : Q_1 = 149,708 kips

Column stress : σ_1 = 10.12 ksi
: σ_2 = 13.52 ksi

3. A three story shear building has masses $m_1 = m_2 = m_3$ = 20,000 kg damping ratio 2% the first natural frequency f_1 = 1.5 HZ and the first mode as shown.
The lowest columns are of unequal length but they have the same **E**. The spectral velocity was established as S_v = 0.25m/S.



For a horizontal seismic excitation calculate:

a) The maximum displacement at the top.

(Displacement $\hat{u}_{ij} = \dfrac{L_j \Phi_{ij}}{M_j} S_d$)

b) Equivalent earthquake forces and base shear.

c) Maximum bending stresses at the heads of the lowest exterior column and the lowest interior column. The depth of the cross section is 482 mm and its $\mathbf{I} = 910 \times 10^6\ \mathrm{mm}^4$

Hint:

Generalized masses $\mathbf{M_1 = m_1\Phi_{11}^2 + m_2\Phi_{21}^2 + m_3\Phi_{31}^2 = Mj = \sum m_i\Phi_{ij}^2}$

$$\mathbf{L_j = \sum m_i \Phi_{ij}}$$

$$\mathbf{L_1 = m_1\Phi_{11} + m_2\Phi_{21} + m_3\Phi_{31}}$$

a) max. top disp = $\mathbf{\hat{u}_{31} = \dfrac{L_1}{M_1}\Phi_{31}S_d}$ $\quad \mathbf{, S_d = \dfrac{S_v}{\omega_1}}$

b) $\mathbf{q_{i1} = m_i\omega_1^2\hat{u}_{i1}}$

$$\mathbf{q_{11} = m_1\omega_1^2\hat{u}_{11}}$$

$$\mathbf{q_{21} = m_2\omega_1^2\hat{u}_{21}}$$

$$\mathbf{q_{31} = m_3\omega_1^2\hat{u}_{31}}$$

Q= base shear = $\mathbf{\sum q_{i1}}$

c) $\mathbf{Q_1 = \dfrac{k_1}{k_1 + k_2 + k_1} Q}$, $\quad \mathbf{Q_2} = \surd$

$$\mathbf{BM_1 = \frac{Q_1}{2} \times 6m}$$

$$\sigma = \mathbf{\frac{BM_1 \cdot y}{I}}$$

CHAPTER (7)

COMPLEX RESPONSE ANAYLSIS

(FOURIER TRANSFORM METHOD)

7.1 INTRODUCTION

The Equations of motion of the structure system, Equation 2.2, can be solved in the frequency domain using the complex response method (Fourier transform method).

The procedure of Fourier anaylsis may be explained by the following three steps; a) the load **P(t)** in the time domain is expressed in terms of harmonic components **P(f)** by using continious or discrete Fourier transformation, b) the response of the structure to each harmonic component is evaluated, c) then these response components are superimposed by using the inverse Fourier transform to obtain the total response of the structure in the time domain.

The continious Fourier transform (Fourier integral) is defined as

$$\mathbf{p(f)} = \int_{-\infty}^{+\infty} \mathbf{P(t)e^{-i2\pi ft}dt} \tag{7.1a}$$

and its inverse is defined as

$$\mathbf{P(t)} = \int_{-\infty}^{+\infty} \mathbf{P(f)e^{+i2\pi ft}df} \tag{7.1b}$$

The infinite time integral of the continious Fourier transform is replaced by a finite sum known as discrete Fourier transform (DFT). The first step in deriving the discrete Fourier transform of the input load is to assume that the loading is periodic of period

$$\mathbf{T_P = N.\ \Delta T} \tag{7.2}$$

where, N=the number of equal time increments **ΔT**. This constitutes an approximation in the treatment of an arbitrary general loading, but is necessary so that the pair given by Equations 7.1 is amenable to digital computation. In the next subchapter, the discrete Fourier transform is derived graphically based on continious Fourier transform theory, and errors due to this approximation are presented with various measures that can be taken to minimize their effects.

7.2 GRAPHICAL DEVELOPMENT OF DISCRETE FOURIER TRANSFORM

Consider the example function **h(t)** and its Fourier transform **H(f)** illustrated in Figure 7.1a. To determine the Fourier transform of **h(t)** by means of digital anaylsis, it is necessary to sample and truncate **h(t)**. Sampling is accomplished by multiplying **h(t)** by the sampling function, Δ_o, in Figure 7.1b. The sample interval is **T**, the sampled function , $\Delta_o(t)$, and its Fourier transform , $\Delta_o(f)$, are illustrated in Figure 7.1b. The modified Fourier transform of the sampled function, $H(f)\times\Delta_o(f)$, (Figure 7.1c), differs from the original continious transform pair by the "aliasing" (over–lapping) effect which results from sampling. To reduce this error we have only one recourse , that is to sample faster , i.e, choose **T** smaller .

The next step is the truncation of the sampled function so that only a finite number of points, say **N**, are considered. The rectangular or truncation function, **X(t)**, and its Fourier transform (**sin f/f** function) are illustrated in Figure 7.1d. The frequency transform of the sampled truncated function has now a "ripple" to it (Figure 7.1e); to reduce this effect it is desirable to choose the length, $\mathbf{T_o}$, of the truncation function **X(t)** as long as possible hence the **sin f / f** function will approach an impulse.

The more closely the **sin f / f** approximates an impulse, the less ripple will be introduced by the convolution which results from truncation. The modified transform pair of Figure 7.1e, $\mathbf{H(f)\times\Delta_o(f)\times X(f)}$, is still not an acceptable discrete Fourier transform pair because the frequency transform is still a continious function. For machine computation, only sample values of the frequency function can be computed, thus it is necessary to modify the frequency transform by the frequency sampling function, $\mathbf{\Delta_1\ (f)}$ illustrated in Figure 7.1f. The frequency sampling interval is $\mathbf{1 / T_o}$.

As illustrated in Figure 7.1g, both the original time function, **h(t)**, and the original Fourier transform, **H(f)**, are approximated by **N** samples. These **N** samples define the discrete Fourier transform pair and approximate the original Fourier transform pair. As shown, sampling in the time domain resulted in a periodic function of frequency (Figure 7.1c), and sampling in the frequency domain resulted in a periodic function of time (Figure 7.1f). Hence, the discrete Fourier transform requires that both the original time and frequency function be modified such that they become periodic.

In summary, the discrete Fourier transform is developed from the continious Fourier transform by sampling and truncating the continious time function assuming it is periodic, and finally sampling the resulting frequency continious function at equidistant points in order to bring it to machine computation. Consequantly, numerical integration of Equation 7.1a implies the relationship

$$\mathbf{P(f_k) = \sum_{i=0}^{N-1} P(t_i)e^{-i2\pi f_k t_i}(t_{i+1} - t_i)} \qquad \mathbf{k = 0,1,\ldots\ldots\ldots\ldots N-1} \tag{7.3}$$

7.3 APPLICATION OF THE FOURIER TRANSFORM METHOD TO DYNAMIC PROBLEMS

The discrete Fourier transform (Equation 7.3) offers a potential method of attack to dynamic problems. However, careful inspection of Equation 7.3 reveals that if there are **N** data points of the load function $\mathbf{P(t_i)}$, and if **N** amplitudes, $\mathbf{P(t_i)}$, are to be determined, then computation time is proportional to $\mathbf{N^2}$, the number of multiplications. Even with high speed computers, computation of the discrete Fourier transform requires exceesive machine time for large **N**. Consequantly, an efficient computational algorithm is developed by Cooley and Tukey. This is known as the fast Fourier transform (FFT), which reduces the computing time of Equation 7.3 to a time proportional to $\mathbf{N\ log_2 N}$. This increase in computing speed has completely revolutionized the field of structural dynamics and has made the frequency domain approach computationally competitive with the traditional time domain anaylsis.

For this purpose, each input force **P(t)** is assumed to be given for an even number, of equidistant points in the time domain as

$$\mathbf{P_k(t) = P(t = k.\Delta T)}, \qquad \mathbf{k = 0,1,\ldots\ldots\ldots\ldots N-1} \tag{7.4}$$

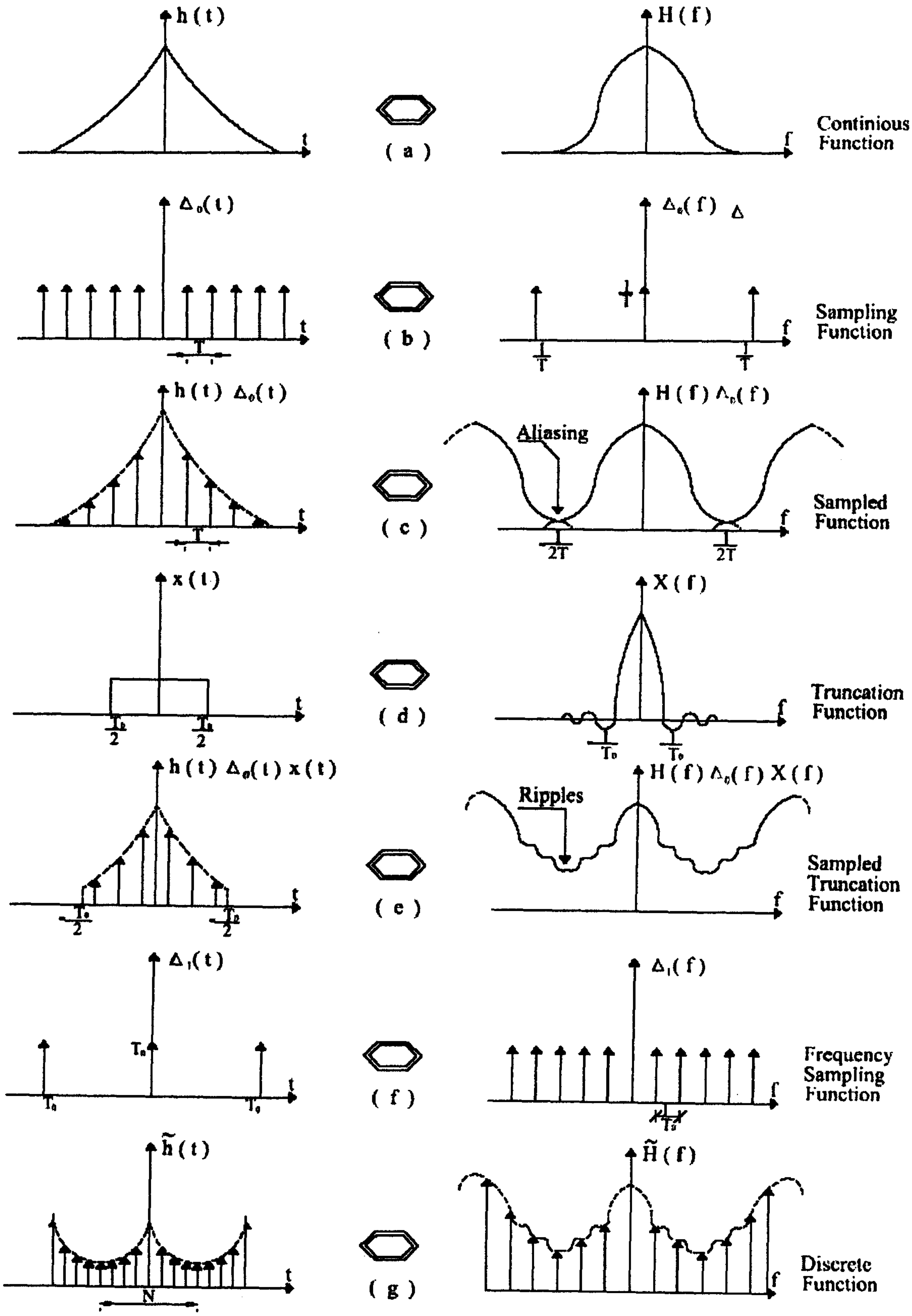


Figure 7.1 Graphical development of the discrete fourier transform

Accordingly, Equation (7.1b) takes the discrete form

$$P_k(t) = \sum_{S=0}^{N-1} P_S \exp(i\frac{2\pi kS}{N}) \qquad k = 0,1,\ldots\ldots\ldots\ldots N-1 \tag{7.5}$$

P_S are complex amplitudes in the frequency domain called Fourier transforms of $P_k(t)$ and are defined as

$$P_S = \frac{1}{N}\sum_{k=0}^{N-1} P_k(t)\exp(-i\frac{2\pi kS}{N}) \qquad S = 0,1,\ldots\ldots\ldots\ldots N-1 \tag{7.6}$$

the power $i\frac{2\pi kS}{N}$ may be written as $i\omega_s t$ where ω_s are frequencies

$$\omega_s = \frac{2\pi S}{N.\Delta T} = \Delta\overline{\omega}.S \tag{7.7}$$

where $\Delta\overline{\omega}$ and $\Delta\overline{\omega}(N-1)$ are the lowest and highest frequencies to be considered in the anyalsis.

The evaluation of the sum in Equation 7.5 will be most efficient if the number of time increments **N** is a power **M** of *2*, that is $N = 2^M$

Then, each of the terms Equation 7.4 is considered as an input to Equation 2.2, i.e.

$$[m]\{\ddot{u}\}_s + [c]\{\dot{u}\}_s + [k]\{u\}_s = \{P_s\}\, e^{i\omega_s t} \tag{7.8}$$

assuming the steady state solution

$$\{u\}_s = \{\overline{u}\}_s e^{i\omega_s t} \tag{7.9}$$

Equation 7.8 reduce to

$$([k] + i\omega_s[c] - \omega_s^2[m])\{\overline{u}\}_s = \{P_s\} \tag{7.10}$$

Equation 7.10 constitutes a set of linear Equations which can be solved for the complex displacement amplitudes $\{\overline{u}\}_s$. Then $\{u\}_s$, the response at each frequency ω_s, follows from Equation 7.9.

The complete response in the time domain follows by superposition of the response components $\{\overline{u}\}_s$, i.e. $\{u\} = \sum_{S=0}^{N-1}\{u\}_S = \sum_{S=0}^{N-1}\{\overline{u}\}_S e^{i\omega_s t}$. Thus the displacement $\{u\}$ in the time domain can be obtained by performing an inverse fast Fourier transform on each of the terms of $\{\overline{u}\}_s$.

In summary, the time history response can be computed through the use of the Fourier transform by:

(1) finding the direct transform of the excitation forces.
(2) Multiplying it by the transfer function ,

$$H(\omega) = ([k] + i\omega_s [c] - \omega_s^2 [m])^{-1} \tag{7.11}$$

(3) obtaining the inverse transfomation of the product.

A frequency solution has some inherent advantages,

(1) frequency dependent impedance functions can be incorporated .
(2) it allows control of the accuracy of the solution within different ranges of frequancies and,
(3) once the transfer fuctions have been computed, it permits change of the excitation force, or its location, without having to repeat the complete procedure.

CHAPTER (8)

RESPONSE TO RANDOM LOADS IN ONE DEGREE OF FREEDOM

A random process differs from the deterministic processes dealt with in the preceding chapter in that it cannot be accurately predicted mathematically even if the past time history is known. Such a process is most meaningfully described in statistical terms. The basic statistical characteristics are reviewed first, presuming that the random process **x (t)** may represent loading of the structure or its response.

A random process may be given in the form of one representative time history (Figure 8.1) or by a set of sample functions collected into an ensemble (Figure 8.2). When the statistical characteristics are extracted from the one long time history, the procedure involved is called temporal averaging. The anyalsis of the ensemble is known as ensemble averaging. Ideally, the time histories should extend from $t \to -\infty$ to $t \to +\infty$

8.1 BASIC STATISTICAL CHARACTERISTICS

Probability density function, P(x). This function defines the probability that **x** will have a value in the range from **x** to **x+dx**. The typical bell-like shape of this function is indicated in Figure 8.3.

$$P(x) = p(x = x_i)$$

Because all values may occur

$$\int_{-\infty}^{+\infty} p(x)dx = 1 \qquad (8.1)$$

Probability distribution function, **P(x)**. This function defines the probability of **x** being smaller than or equal to a certain value **a** and thus

$$P(x) = Pr(x \le a) = \int_{-\infty}^{a} p(x)dx \qquad (8.2)$$

note that: $\int_{-\infty}^{\infty} p(x)dx = \text{area} = 1$

this function is characterized by an S-like shape and is always bounded by the limits $0<P(x) < 1$ (Figure 8.4)

Mean value or expected value. This is the mean or average value of the function and can be defined for the function **x(t)** as

$$\bar{x} = \frac{1}{\underset{T\to\infty}{2T}} \int_{-T}^{+T} x(t)dt \qquad (8.3)$$

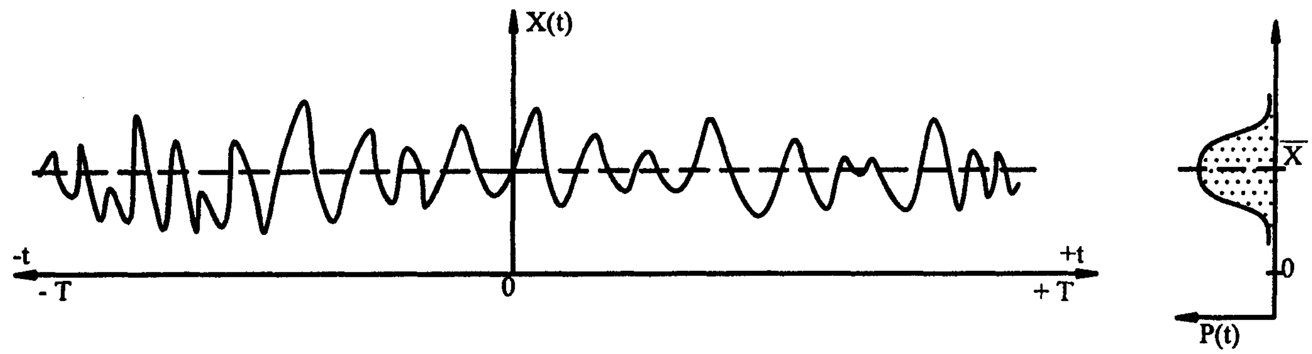


Figure 8.1 A random function.

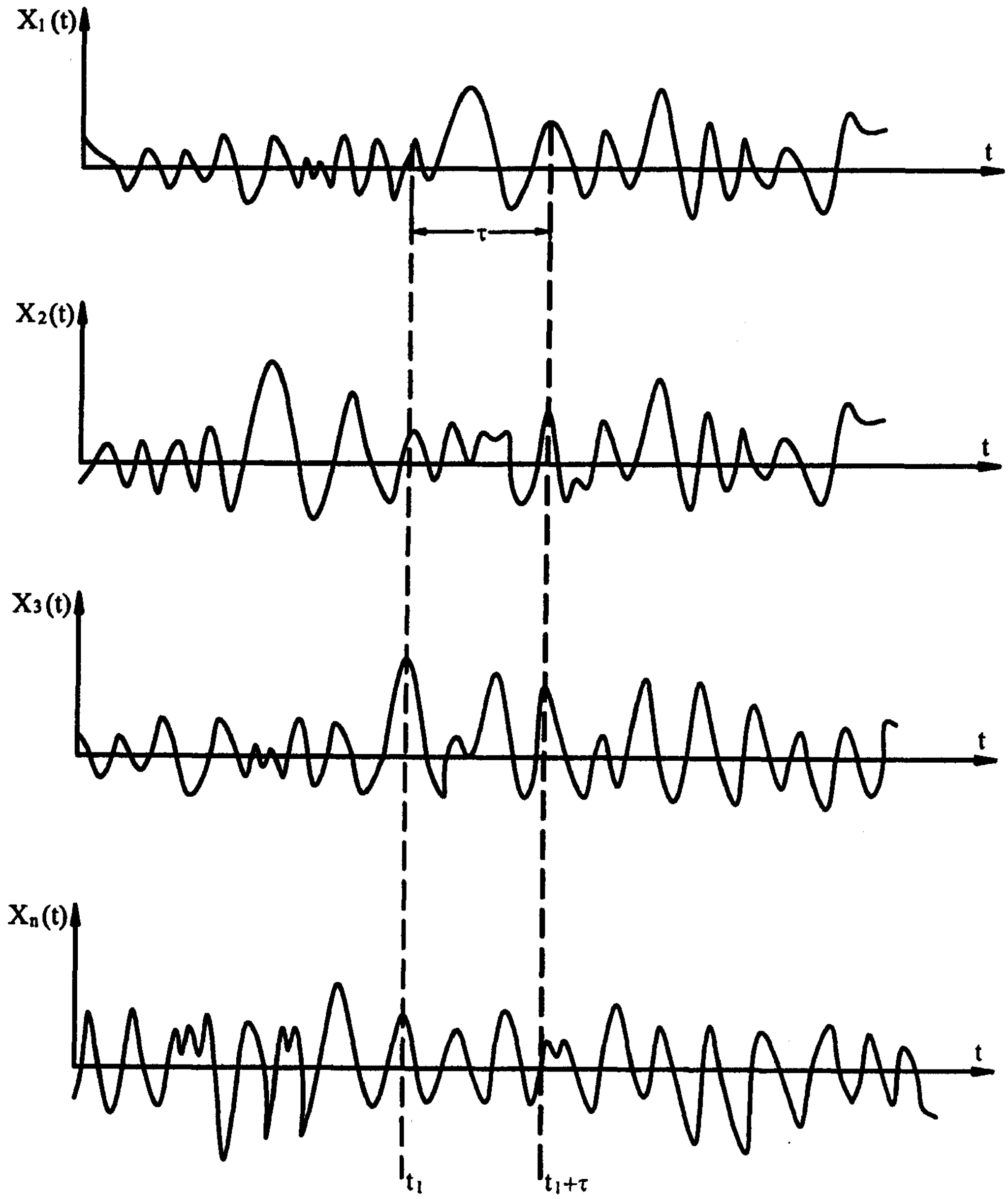


Figure 8.2 Ensemble of samples of random function.

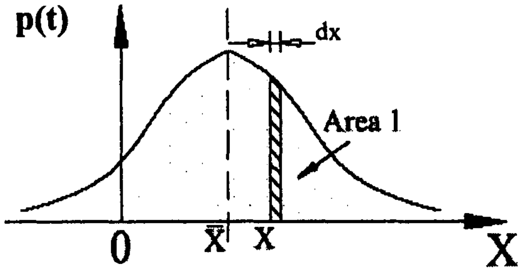


Figure 8.3 **Probability density.**
or **Probability mass function.**
or **Probability density function.**

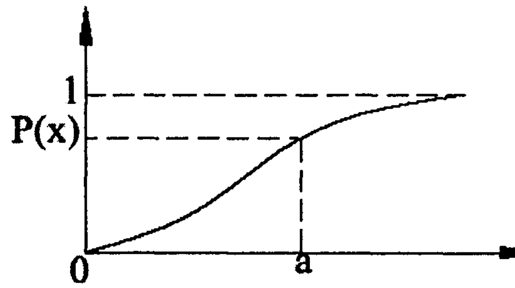


Figure 8.4 **Probability distribution.**

Realizing that all values have a total probability of occurring equal to 1, the mean value can also be expressed using the probability density function as

$$\bar{x} = \int_{-\infty}^{+\infty} x p(x) dx \tag{8.4}$$

Thus, the value $\bar{x}$ can be viewed as the coordinate of the centroid of the area under the curve $p(x)$.

Finally, the mean value can be obtained from the ensemble of samples by cutting through the ensemble at a certain time, e.g. t_1 and calculating the average value of the samples (Figure 8.2). This gives

$$\bar{x} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i(t_1) \tag{8.5}$$

other notations used are $\bar{x} = E(x) = \langle x \rangle$.

Mean-square value is the average value of $x^2(t)$, is denoted as $\overline{x^2} = E(x^2) = \langle x^2 \rangle$ and follows from the time history of $x(t)$ as

$$\overline{x^2} = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt \quad T \to \infty \tag{8.6}$$

or from the ensemble as

$$\overline{x^2} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i^2(t_1) \tag{8.7}$$

Variance is the mean-square value of the difference from the mean. Its notations are $\sigma_x^2 = \overline{(x - \bar{x})^2}$ and its value is

$$\sigma_x^2 = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}(x-\bar{x})^2 dt \tag{8.8}$$

or from the probability density function

$$\sigma_x^2 = \int_{-\infty}^{\infty}(x-\bar{x})^2 p(x)dt \tag{8.9}$$

Small variance indicates less widely spread distribution about the mean.

Standard deviation is the square root of variance and is, therefore, also called the root-mean-square value of **x**, or briefly r.m.s. Standard deviation is usually denoted as

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{\overline{(x-\bar{x})^2}} \tag{8.10}$$

with regard to equation 8.1 and 8.9, the standard deviation can be viewed as the radius of gyration of **p(x)** about $\bar{x}$.

Standard deviation is an important magnitude and together with the mean value $\bar{x}$ are the most important parameters that can characterize a probability distribution function.

$$\text{mean} \Rightarrow \text{center of gravity} = \frac{\int x p(x)dx}{\int p(x)dx = 1} = \frac{\text{Statical moment of area}}{\text{area}}$$

also $$\text{Variance} \Rightarrow \text{moment of Inertia} = \frac{\int (x-\bar{x})^2 p(x)dx}{\int p(x)dx} = \frac{\text{moment of Inertia}}{\text{area}}$$

Gaussian or normal probability density and distribution function are defined as

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{\frac{-(x-\bar{x})^2}{2\sigma_x^2}} \tag{8.11}$$

and

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma_x}\int_{-\infty}^{x} e^{\frac{-(x-\bar{x})^2}{2\sigma_x^2}} dx \tag{8.12}$$

This distribution is called Normal because it fits most natural phenomena. It can be shown that a force described by a normal distribution produces response of linear systems which also has a normal distribution.

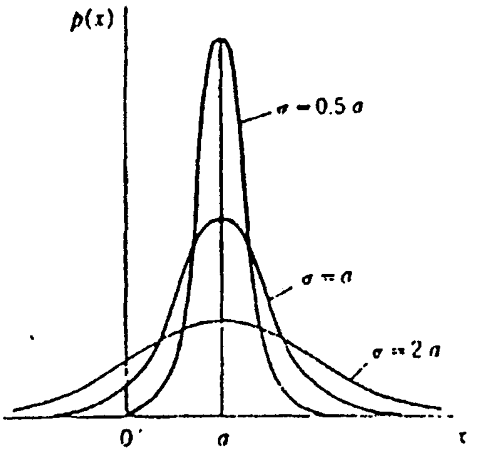


Figure 8.5 Normal distributions with $\bar{x}$ =a and different values of σ.

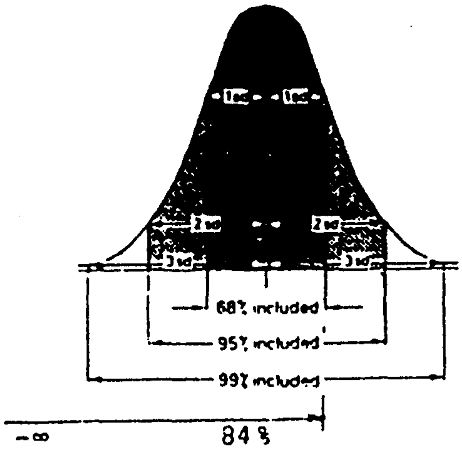


Figure 8.6 Probabilities associated with multiples of standard deviation σ=sd.

The normal distributions is defined by the mean value, $\bar{x}$, and standard $\sigma_x = \sigma$. The magnitude of σ indicates the spread of the values of **x** about the mean (Figure 8.5). The dimensionless measure of this spread is the ***coefficient of variation*** defined as the ratio $\sigma/\bar{x}$

The normal distribution is widely used and various probabilities following from it are tabulated. For example, the probability of **x** being smaller than or equal to the mean plus one standard deviation is 84%, i.e.

$$P(x \le \bar{x} + \sigma) = 84\%$$

this probability characterizes the pseudovelocity spectra plotted in Figure 5.3. Other probabilities are indicated in Figure 8.6.

A given probability density of the process describes the percentage (proportion) of time for which **x** takes on values in a certain range. However, it does not provide any information on the rate of change in **x(t)** i.e, on the frequency characteristics of the process. A more Complete description of a random process is contained in further statistical characteristics called ***correlation functions*** and ***power spectral densities.***

Autocorrelation function or more briefly correlation function is defined as the mean value of the product of **x(t)** and **x(t + τ)** where τ is a time lag, i.e.

$$R_x(\tau) = \overline{x(t)x(t+\tau)} = \frac{1}{2T}_{T\to\infty} \int_{-T}^{T} x(t)x(t+\tau)dt \tag{8.13a}$$

From the ensemble (Figure 8.2), the correlation function is obtained as

$$R_x(\tau) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i(t_1)x_i(t_1+\tau) \qquad (8.13b)$$

By evaluating the average products for different values of τ, the correlation function of a process is established. The correlation function is even, droops either in a smooth or oscillatory way and has the following properties:

$$R(0) = \overline{x^2} \quad , \quad R(\infty) = 0$$

$$R(\tau) = R(-\tau) \qquad \frac{d}{d\tau}R(0) = 0 \quad \text{(horizontal tangent)}$$

Since τ is real time in seconds, the correlation function indicates the speed with which the correlation of the process diminishes and the time within which it vanishes (Figure 8.7).

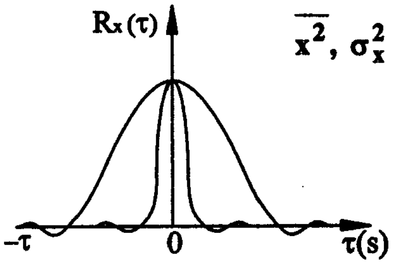


Figure 8.7 Typical autocorrelation functions.

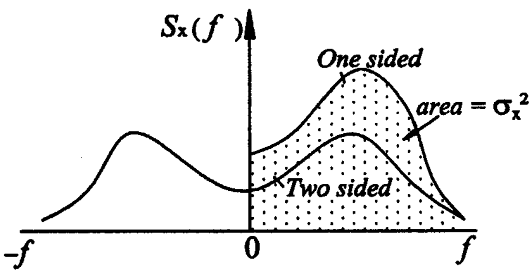


Figure 8.8 Two sided and one sided power spectral densities.

The above statistical characteristics allow a few more definitions:-

If the statistical characteristics are independent of the reference time t_1 where we make the cut (Figure 8.2) the process is called ***stationary***; if they depend on time t_1 the process is ***nonstyationary***. The discussion here is limited to stationary processes. If the process is stationary and the temporal averages are equal to the ensemble averages, the process is ***ergodic***.

A **centric** process is a stationary process with $\bar{x}=0$

Covariance is one of the measures of the extent to which two random variables **x(t)** and **y(t)** are related to each other or correlated. Covariance is defined as

$$\sigma_{xy}^2 = \overline{x(t)y(t)} = \langle x(t)y(t) \rangle \qquad (8.14)$$

when **x(t)** and **y(t)** are completely independent $\sigma_{xy}^2 = 0$; σ_{xy}^2 indicates correlation between the two variables, e.g. input force and response.

In stationary processes the mean $\bar{x}$ is constant and it is, therefore, in favour of numerical accuracy to separate the mean value from the process and analyze just the fluctuating random part of it, $x'(t)=x(t)-\bar{x}$. If $x(t)$ is a load, $\bar{x}$ is its static component which produces static deflection about which the structure oscillates due to the effect of the fluctuating component $x'(t)$. Thus, the statistical analysis can be limited to the fluctuating component because $R_x(\tau) = R_{x'}(t) + \bar{x}^2$.

Power spectral density, **S(f)**. This function describes the energy distribution of the process with regard to frequency and is defined as the Fourier transform of the function $R_x(\tau)$, i.e.

$$S(f) = \int_{\infty}^{\infty} R(\tau)e^{-i2\pi f\tau}d\tau \tag{8.15}$$

This transformation yields an even, two-sided power spectrum indicated in Figure 8.8. Because negative frequencies do not have a technical meaning, it is usually preferable to define a one-sided power spectrum for positive frequencies only. The area under both types of the spectra has to be the same and thus, the magnitude of the one-sided spectrum is twice the magnitude of the two-sided spectrum (Figure 8.8). Splitting the integration interval into two, -∞ to 0 and 0to+∞ and recalling that $R(-\tau) = R(\tau)$, the Fourier transform reduces to a cosine Fourier transform and the one-sided power spectrum becomes

$$S(f) = 4\int_0^{\infty} R(\tau)\cos 2\pi f\tau d\tau \tag{8.16}$$

The inverse Fourier transform of the spectrum yields the correlation function.

$$R(\tau) = \int_0^{\infty} S(f)\cos 2\pi f\tau df \tag{8.17}$$

The Fourier transform pair defined by equations 8.16 and 8.17 is also known as the Wiener-Khintchin relationship.

$$R(\tau) \Leftrightarrow S(f)$$

Fourier transform pair

The term power spectrum stems from electrical applications in which it has the following physical meaning. Assume that $x(t)$ is random voltage filtered through a narrow band filter and that the power passed by the filter is measured. Then, this power is proportional to the bandwidth of the filter and the spectral density of $x(t)$ at the centre frequency of the filter. The total power describes the variance of the signal and thus,

$$\sigma^2 = \overline{x^2} = \int_0^{\infty} S_x(f)df = \int_0^{\infty} S_x(\omega)d\omega \tag{8.18}$$

Expressing $d\omega = 2\pi df$, equation 8.18 gives

$$S(f) = 2\pi S(\omega) \tag{8.19}$$

Equation 8.18 defines the most important property of the spectrum and also suggests the dimension of a spectrum because **S(f)df** must have a dimension of $\mathbf{x^2}$. Consequently, $\mathbf{S_x(f)}$ is in (dimensions of $\mathbf{x^2}$)/frequency. Thus if **x** is displacement, $\mathbf{S_x(f)}$ is in $m^2/s^{-1} = m^2s$; the power spectrum of acceleration is similarly $m^2/s^4/s^{-1}=m^2s^{-3}$.

Other forms of power spectra used are the normalized spectrum and the logarithmic spectrum.

Normalized spectrum, $\mathbf{S'(f)}$ is defined **dS** $\mathbf{S'(f) = S(f)/\sigma^2}$ [sec] and thus $\mathbf{S(f) = \sigma^2 S'(f)}$ and

$$\int_0^\infty \mathbf{S'(f)df = 1} \tag{8.20}$$

Logarithmic spectrum is $\mathbf{S^\ell = fS(f)/\sigma^2}$, it is dimensionless and, because d $\log_e$ $\mathbf{f = \frac{1}{f}df}$,

$$\int_0^\infty \mathbf{\frac{fS(f)}{\sigma^2} d\log_e f = 1} \tag{8.21}$$

The relation between $\mathbf{S'(f)}$ and $\mathbf{S^\ell(f)}$ is shown in Figure 8.9.

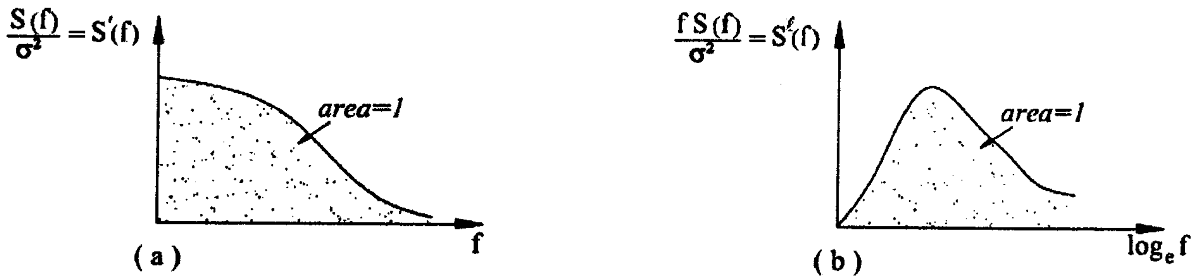


Figure 8.9 Relationship between (a) normalized spectrum and (b) logarithmic spectrum.

When the process $\mathbf{x(t) = const.\ x'(t)}$, the spectrum is, with regard to equations 8.13 and 8.15

$$\mathbf{S_x(f) = (const.)^2 S_{x'}(f)} \tag{8.21a}$$

e.g., the effective earthquake force $\mathbf{P(t) = -m\ddot{y}_g(t)}$ and its spectrum

$$\mathbf{S_p(f) = m^2 S_{\ddot{y}g}(f)} \tag{8.22}$$

where, $\mathbf{S_{\ddot{y}g}(f)}$ is the spectrum of ground acceleration.

Examples of random processes. Examples of typlical random process and their comparison with a deterministic harmonic process are shown in Figure 8.10. Mathematical expressions for some correlation functions and the corresponding power spectra can be found in Reference 16. Power spectra of a few earthquake ground motions are plotted in Figure 8.11.

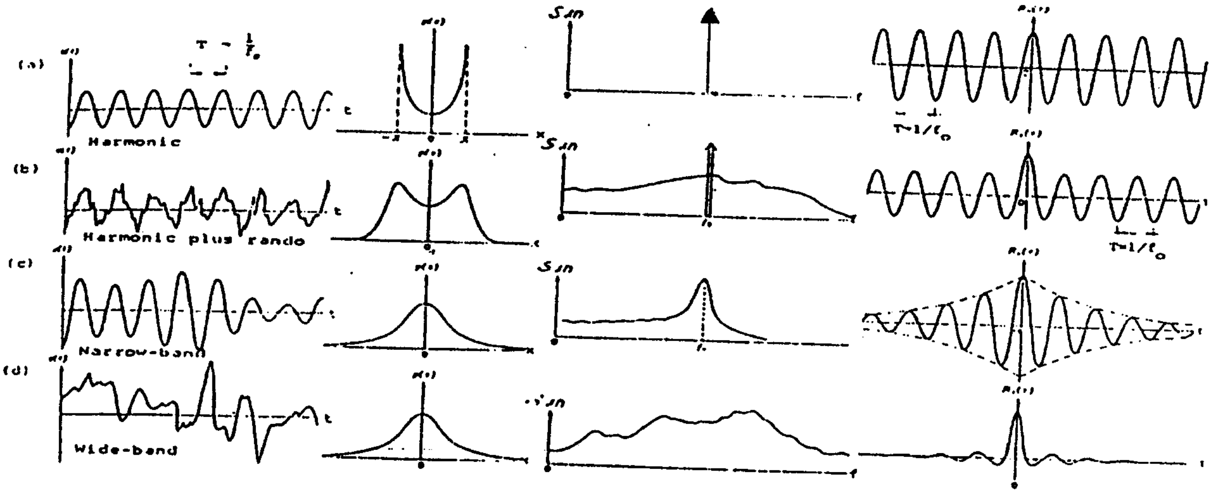
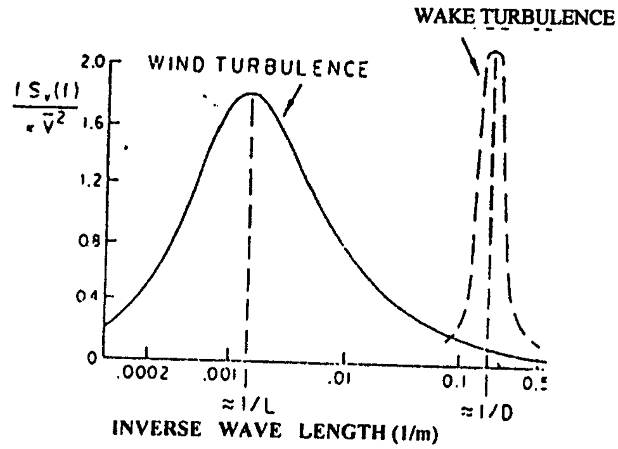


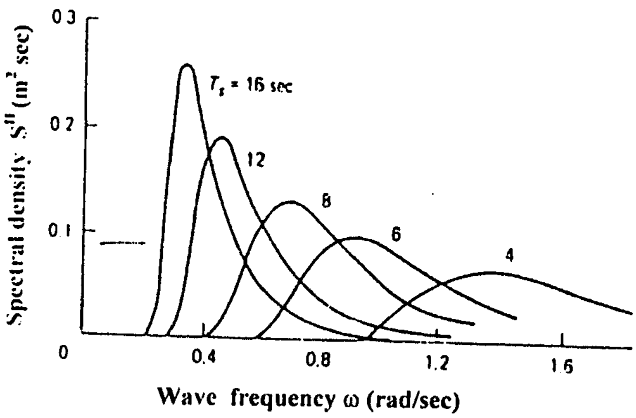
Figure 8.10 **Typical random processes and their description in terms of Probability, power spectral density and autocorrelation function.**



$$S_V(f) = 4K\bar{V}_{10}\frac{L/\bar{V}_{10}}{(2+\bar{f}^2)^{5/6}}$$

K= surface roughness factor=0.08 to 0.14, L=scale length ≈ 4000 ft, $\bar{V}_{10}$=mean wind velocity 10 m above ground in ft/s and $\bar{f} = fL/\bar{V}_{10}$

Figure 8.10a Power spectral density of wind velocity fluctuations Sv(f) and schematic of fluctuations in wake (Davenport, Harris)



$$S = \frac{A}{\omega^5}e^{-B\omega^{-4}}, \quad A = 0.169H_s^2(\frac{2\pi}{T_s})^4,$$

$$B = 0.675(\frac{2\pi}{T_s})^4$$

H_s (in ft)= the significant wave height $\approx 1.6\bar{H}$

T_s (in sec)= the corresponding significant wave period $\approx 1.1\bar{T}$

Figure 8.10b Power spectral density of sea wave displacement for unit significant wave height (Bretschneider, 1959)

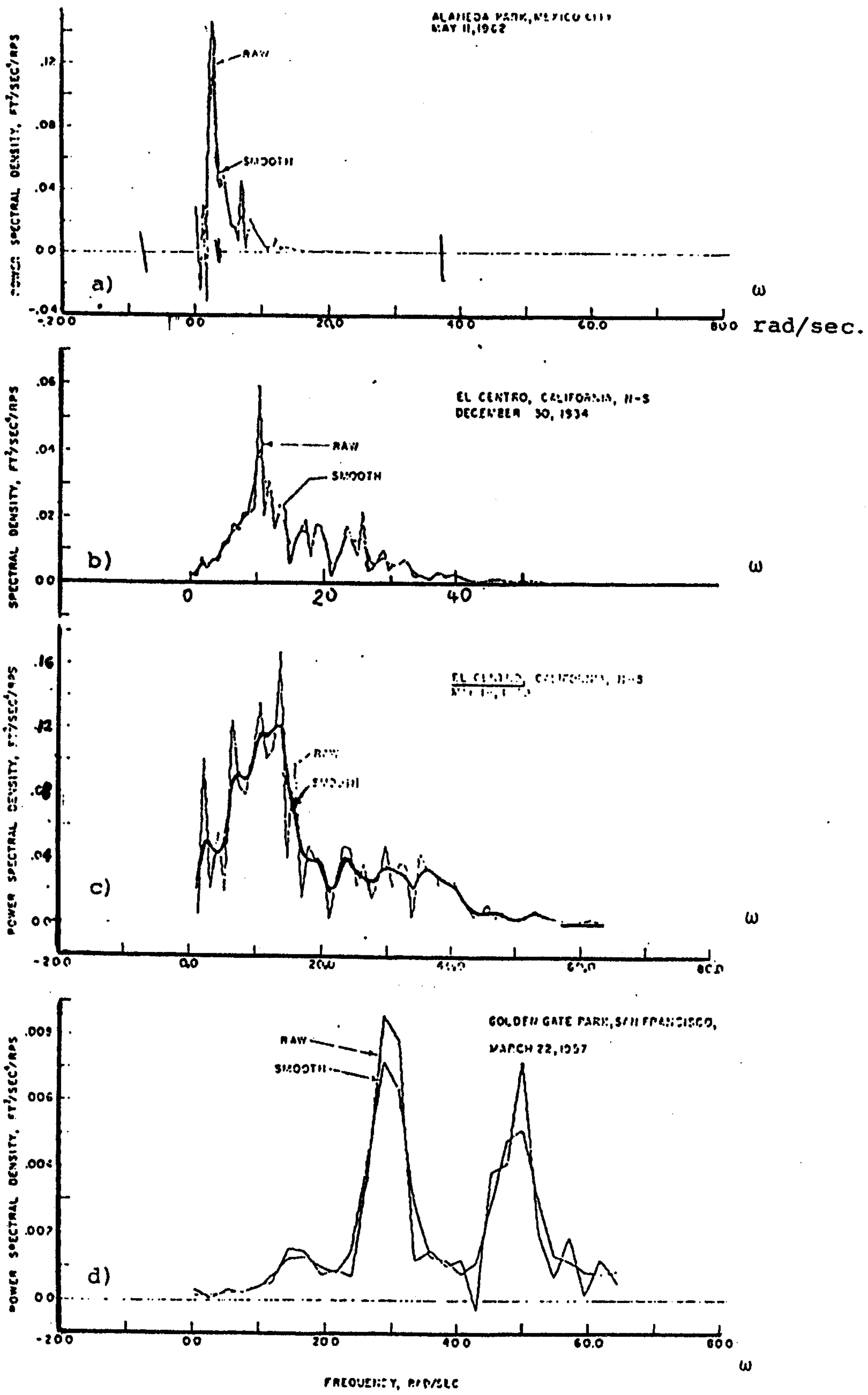


Figure 8.11 Spectra of Earthquake Ground Acceleration.

8.2 RESPONSE TO RANDOM LOAD IN ONE DEGREE OF FREEDOM (RELATION BETWEEN INPUT AND OUTPUT)

A periodic force **P(t)** with period $T = \frac{1}{f_1}$ can be represented by a complex Fourier series as

$$P(t) = \sum_{-\infty}^{\infty} c_r e^{ir2\pi f_1 t}, \quad c_r = \frac{1}{T}\int_{-T/2}^{T/2} P(t) e^{-ir2\pi f_1 t} dt \qquad r = 1,2,\ldots \qquad (8.23)$$

The response of a SDF system to such a load can be obtained by means of superposition of responses to individual components **r** in terms of frequency response function (admittance). The harmonic load

$$P(t) = P_0 e^{i\omega t}$$

Yields response

$$y(t) = \frac{P_o}{k} H(\omega) e^{i\omega t}$$

where

$$H(f) = \frac{1}{1 - (\frac{f}{f_0})^2 + i2D\frac{f}{f_0}} \qquad (8.24)$$

However response to a series of harmonic loads (Equation 8.23)

$$y(t) = \sum_{-\infty}^{\infty} c_r H(f_r) e^{ir2\pi f_1 t} \qquad \text{with } f_r = r\, f_1 \qquad (8.25)$$

The ***mean square*** response can be expressed in terms of Parceval's theorem

$$\overline{x^2} = \sum_{-\infty}^{\infty} |c_r|^2 \qquad (8.26)$$

because (8.25) is again a Fourier series with amplitudes $\frac{1}{k} c_r H(f_r)$. Hence,

$$\overline{y^2} = \frac{1}{k^2} \sum_{-\infty}^{\infty} |c_r|^2 H|f_r|^2 \qquad (8.27)$$

realizing that

$$|x_1 x_2|^2 = |x_1|^2 |x_2|^2$$

A ***non-periodic*** (random) force can only be expressed in the above manner if period **T** is extended to ∞, thus from (8.27)

$$\overline{y^2} = \frac{1}{k^2} \lim_{T\to\infty} 2\sum_0^{\infty} |c_r|^2 |H(f_r)|^2 \tag{8.28}$$

as $|c_r|^2, |H(f_r)|^2$ are even function. Substitution for c_r gives

$$\overline{y^2} = \frac{1}{k^2} \lim_{T\to\infty} \sum_0^{\infty} \frac{2}{T^2} \left| \int_{-T/2}^{T/2} P(t) e^{-ir2\pi f_1 t} dt \right|^2 |H(f_r)|^2$$

with period $T \to \infty$

$$1/T \to df, \sum_0^{\infty} \to \infty \int_0^{\infty}, \int_{-T/2}^{T/2} P(t) e^{-ir2\pi f_1 t} dt \to A(if) \text{ as } rf_1 = f_r \to f \text{ and } H(fr) \to H(f).$$

Also, the mean square response can be expressed by means of its power spectrum, $\overline{y^2} = \int_0^{\infty} S_y(f) df = \sigma_y^2$. Hence

$$\sigma_y^2 = \overline{y^2} = \int_0^{\infty} S_y(f) = \frac{1}{k^2} \int_0^{\infty} \lim_{T\to\infty} \left[\frac{2}{T} |A(if)^2| \right] |H(f)|^2 df$$

As,

$$\lim_{T\to\infty} \frac{2}{T} |A(if)|^2 = S_p(f)$$

i.e. spectrum of the excitation **P(t)**, the relation between spectrum of input and spectrum of output is

$$\boxed{S_y(f) = \frac{1}{k^2} S_p(f) |H(f)|^2 = |\alpha(if)|^2 S_p(f)} \tag{8.29}$$

in which $|H(f)|^2$ is the square of the modulus of the admittance function which is equal to the square of the dynamic magnification factor and is

$$|H(f)|^2 = M^2 = \frac{1}{[1-(f/f_0)^2]^2 + 4D^2 (f/f_0)^2} \tag{8.30}$$

The relation between the input and output described by equation 8.29 is shown in Figure 8.12.

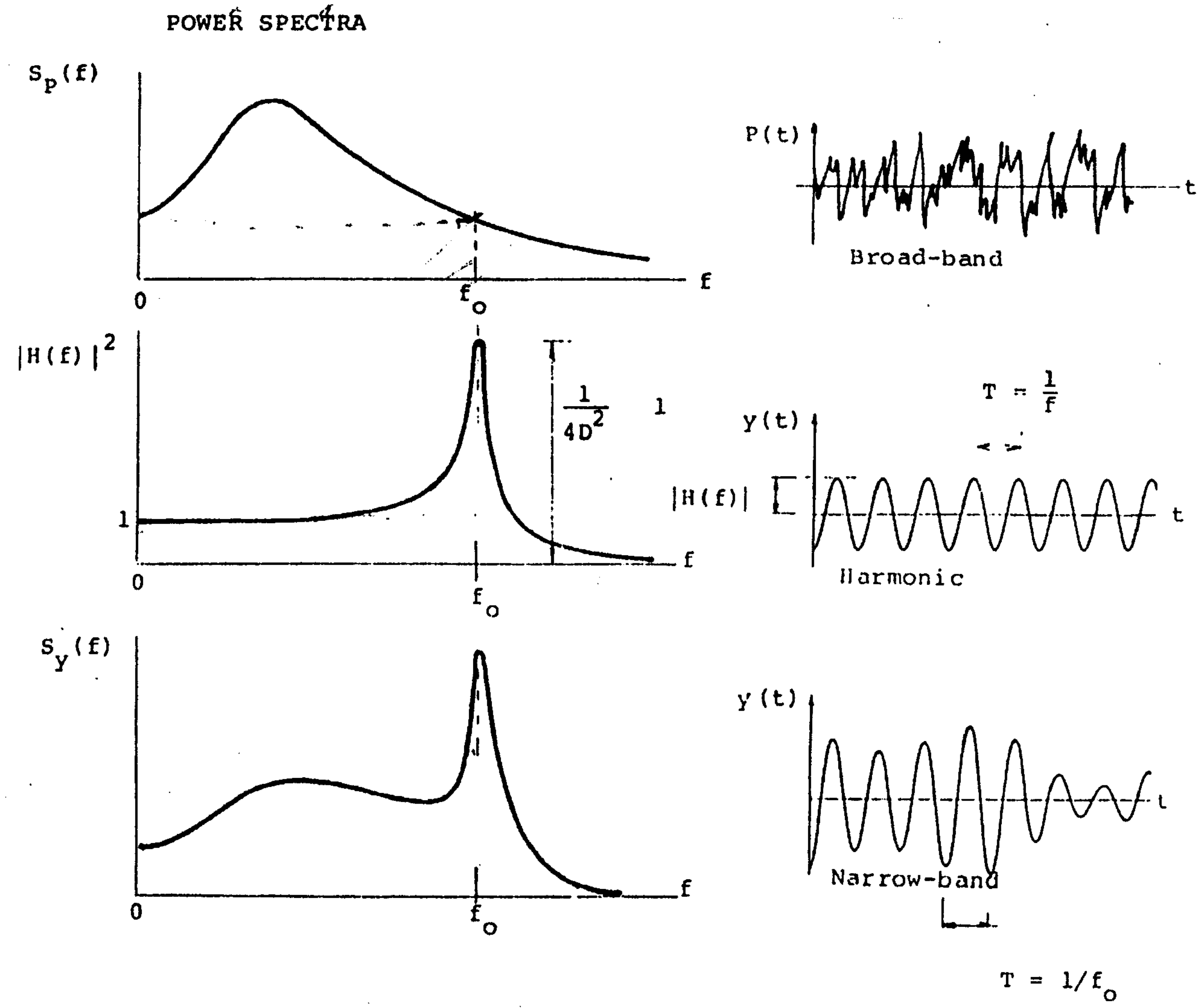


Figure 8.12 The relationship between spectrum of input and spectrum of output.

With response spectrum defined by equation 8.21, the mean square response $\overline{y^2}$ is

$$\sigma_y^2 = \overline{y^2} = \int_0^\infty S_y(f)df = \frac{1}{k^2}\int_0^\infty S_p(f)|H(f)|^2 df \tag{8.31}$$

or in terms of circular frequency

$$\sigma_y^2 = \overline{y^2} = \int_0^\infty 2\pi S_y(\omega)\frac{d\omega}{2\pi} = \int_0^\infty S_y(\omega)d\omega$$

From equation 8.31 the rms displacement having the dimension of amplitudes, is $\sqrt{\overline{y^2}} = \sigma_y$.

The only complication is that the integral in equation 8.31 cannot be generally evaluated in closed form. It can be evaluated approximately as

$$\overline{y^2} \cong \frac{1}{k^2}\int_0^{f_0} S_p(f)df + \frac{1}{k^2}\int_0^{\infty} S_p(f_0)|H(f)|^2 df$$

$$= \frac{1}{k^2}\int_0^{f_0} S_p(f)df + \frac{1}{k^2} S_p(f_0)\frac{\pi f_0}{4D} \tag{8.32}$$

This approximate evaluation is based on replacing $|H(f)|^2$ by unity for frequencies from 0 to f_0 and on replacing the force power spectrum $S_p(f)$ by a constant (white) spectrum $S_p(f_0)$ whose magnitude is equal to the force spectrum for the natural frequency of the system, f_0 (Figure 8.13). The first part of equation 8.32 is called the background effect and the second part the resonant effect.

If greater accuracy is needed, the integral in equation 8.31 can be evaluated using the theory of residual or numerical integration.

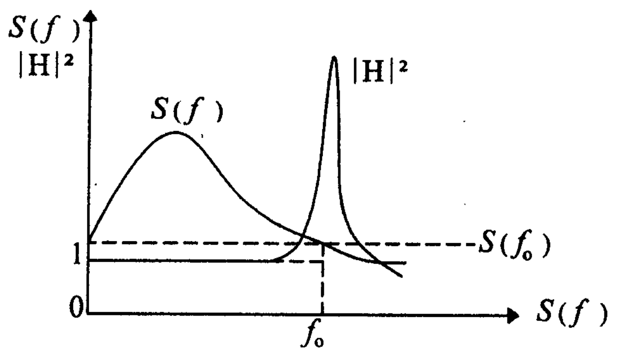


Figure 8.13 Approximate evaluation of response.

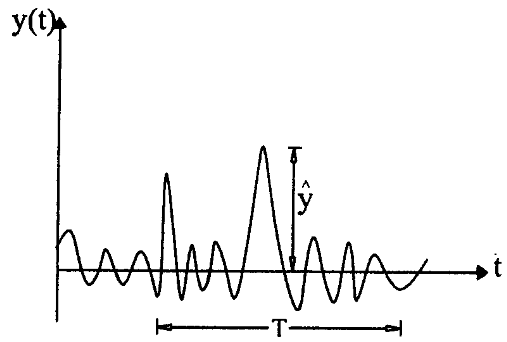


Figure 8.14 Peak value of random response.

When the damping is small and the spectrum is flat, the second part of equation 8.32 yields sufficient accuracy and thus, the variance of the response is approximately

$$\overline{y^2} \cong \frac{1}{k^2} S_p(f_0)\frac{\pi}{4}\frac{f_0}{D} = \frac{1}{k^2} 2\pi S_p(\omega_o)\frac{\pi}{4}\frac{\omega_0}{2\pi D} = \frac{1}{k^2} S_p(\omega_o)\frac{\pi}{4}\frac{\omega_0}{D} \tag{8.33}$$

The input spectrum and the system natural frequency can be expressed in terms of frequency f (Hz) or ω (rad/sec). In both cases the formulae are formally the same.

From the variance $\overline{y^2}$ the standard deviation (root mean-square) of the response follows as

$$\sigma_y = \overline{y^2}$$

The r.m.s response depends on the square root of the damping ratio.

The standard deviation determines the distribution of all values of the response, as can be seen from equation 8.11 and Figure 8.6, but the peak, i.e. maximum value of the response indicated in Figure 8.14 is of primary importance for design.

Peak Value of Response

During each period of observation **T** one largest (peak) value of the response can be established. This largest value depends on the duration of the observation, T, and the apparent frequency, ν, which depends on the spectrum of the process and is (16)

$$\nu = \sqrt{\frac{\int_0^\infty f^2 S(f)df}{\int_0^\infty S(f)df}} \quad \text{(Hz)} \tag{8.34}$$

For a narrow band process such as the response of lightly damped system, the apparent frequency ν is close to the natural frequency and thus, $\nu \approx f_0$. The peak values observed in individual observations may be assembled to yield a probability density distribution (Figure 8.15). The mean value of the peaks can be evaluated as

$$\bar{\hat{y}} = g\sigma_y \tag{8.35}$$

in which the peak factor $g = \bar{\hat{y}} / \sigma_y$ can be calculated using the formula (7)

$$g = \sqrt{2\log_e \nu T} + \frac{0.5772}{\sqrt{2\log_e \nu T}} \tag{8.36}$$

The peak factor ranges between about 2.5 and 4.5 (Figure 8.16)

Response to earthquakes. With regard to equation 8.22 and 8.33, the variance of earthquake response is

$$\overline{y^2} = \frac{1}{k^2}\frac{\pi}{4}\frac{\omega_0}{D} m^2 S_{\ddot{y}_g}(\omega_0) = \frac{\pi}{4}\frac{1}{D\omega_0^3} S_{\ddot{y}_g}(\omega_0) \tag{8.37}$$

A more accurate analysis should consider nonstationarity but the assumption of stationarity is conservative.

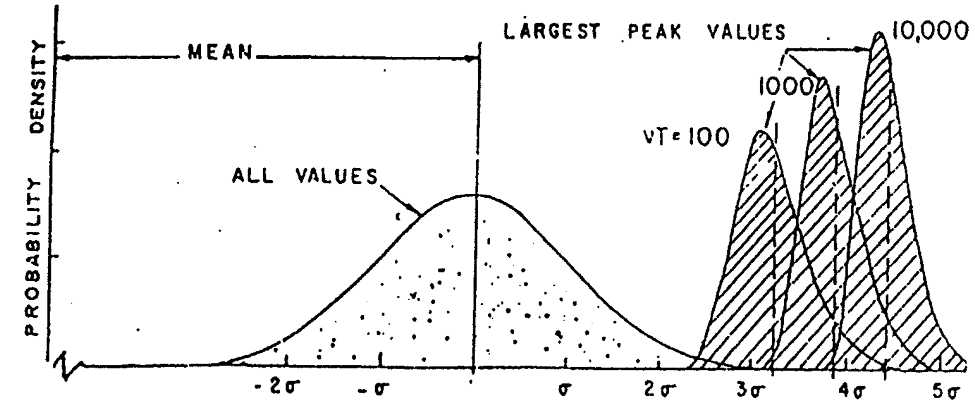


Figure 8.15 Probability distributions of all values and peak values.

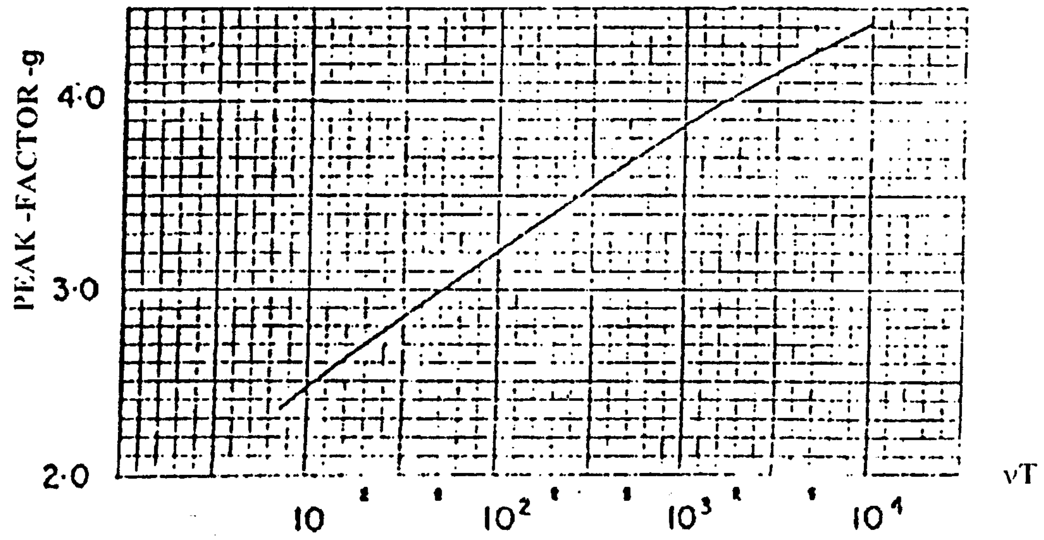


Figure 8.16 Peak factor versus νT

Problem:

Predict the seismic response of the one story shear building given in problem 3.3 to the El-Centro 1940 earthquake in terms of random vibration. The power spectrum of that earthquake is given in Figure 8.11c. Assume damping ratio = 2% and strong motion duration **T** = 30 s.

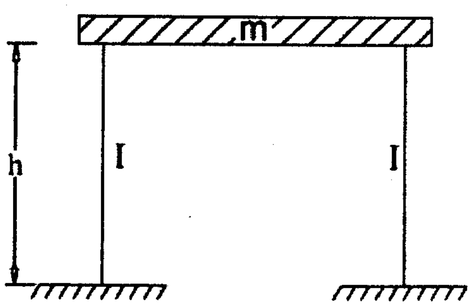
m = 30,000 Kg

I = 560×10^6 mm^4

E = 2×10^5 Mpa

H = 5 m

k = 2×12 EI/h^3



CHAPTER (9)

RESPONSE TO GUSTING WIND

9.1 INTRODUCTION

Assume that the structure is smaller than the typical dimension of the atmospheric vortex. Then the quassi-steady approach applies and the time variable drag force is

$$F_D(t)=\frac{1}{2}\rho C_D A V^2(t)=\frac{1}{2}\rho C_D A[\overline{V}+v(t)]^2 \tag{9.1}$$

Where, ρ=air density, C_D=drag coefficient, A=area exposed to wind, $V(t)=\overline{V}+v(t)$= wind speed with $\overline{V}>v(t)$. Taking the mean wind speed $\overline{V}$ in front the bracket

$$V^2(t)=\overline{V}^2[1+\frac{v(t)}{\overline{V}}]^2=\overline{V}^2[1+2\frac{v(t)}{\overline{V}}+(\frac{v(t)}{\overline{V}})^2]$$

Because $v(t)/\overline{V}<1$ and thus $(v(t)/\overline{V})^2<<1$, the square of the ratio $v(t)/\overline{V}$ can be omitted and the total drag force split into two parts:

The mean (static drag) $$\overline{F_D}=\frac{1}{2}\rho C_D A\overline{V}^2 \tag{9.2}$$

and

The fluctuating (dynamic) drag $F_D(t)=\rho C_D A\overline{V}v(t)$ (9.3)

In which v(t)=fluctuating component of the wind speed. This fluctuating wind speed has a spectrum $S_v(f)$ and, therefore, the spectrum of the drag is, according to Equation 8.21a,

$$S_F=(\rho C_D A\overline{V})^2 S_V(f) \tag{9.4}$$

or

$$S_F=\frac{4\overline{F}_D^2}{\overline{V}^2}S_V(f) \tag{9.5}$$

The relation between the size of the structure and the size of the vortex (disturbance) can be introduced through an "aerodynamic admittance function" analogous to the mechanical admittance,

$$X^2=\left|X(\frac{fL}{\overline{V}})\right|^2 \tag{9.6}$$

where L=characteristic dimension for buildings or "point" structures equal to $L=\sqrt{A}$. The ratio $fL/\overline{V}$ is dimensionless frequency equal to the inverse value of Strouhal number. Only with "point" (small) structures $X\cong 1$. Otherwise the values of X range between these limits:

$$(fL/\overline{V}) \longrightarrow 0 \quad , \quad X^2 \to 1$$
$$(fL/\overline{V}) \longrightarrow \infty \quad , \quad X^2 \to 0$$

X^2 is a dropping function indicated in Figure 9.1.

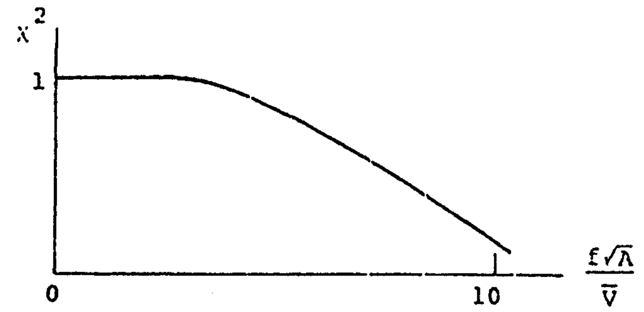


Figure 9.1 Aerodynamic admittance function.

The aerodynamic admittance function is usually established analytically using experimental data. With the aerodynamic admittance introduced into Equation 9.5, the fluctuating force spectrum is

$$S_F = \frac{4\overline{F}_D^2}{\overline{V}^2} \left| X\left(\frac{fL}{\overline{V}}\right) \right|^2 S_V(f) \tag{9.7}$$

with the spectrum given by Equation 9.7, the response of a single degree of freedom structure can be readily predicted. However, for large structures, the approach must be somewhat extended to include the variation of wind speed with the height and the shape of the vibration mode, these aspects were included in an approach adopted in many codes; the approach is called the gust factor approach.

9.2 GUST FACTOR APRROACH

The gust factor approach is a design procedure derived on the basis of the random vibration theory by means of a few simplifying assumptions. It considers only the reponse in the first vibration mode, which is assumed to be linear. These assumptions are particularly suitable for buildings. The method yields all data needed in design: the maximum response, the equivalent static wind load that produce the same maximum response, and the maximum acceleration needed for the evaluation of the physiological effects of strong winds (human comfort).

The gust factor G is, as defined by Davenport, the ratio of the expected peak displacement (load) in a period T to the mean displacement (load) $\overline{u}$. It accounts for the dynamic effect of gusting wind as shown in Figure 9.2. Hence, maximum expected response

$$u_{max} = G\overline{u} = (1 + g\frac{\sigma_u}{\overline{u}})\overline{u} \tag{9.8}$$

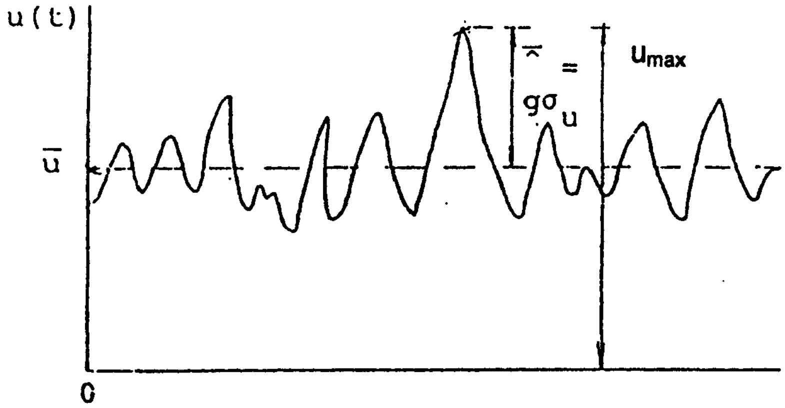


Figure 9.2 Total structural response to gusting factor.

similar approaches have been proposed others but these can all be reduced to the same form of Equation 9.8. This gives the gust factor **G** as

$$G = 1 + g\sqrt{\frac{K}{C_e}\left(B + \frac{SF}{D^t}\right)} \tag{9.9}$$

where, **g**=peak factor, K=roughness factor, C_e=exposure factor, **B**= the background turbulence effect and $\frac{SF}{D^t}$ the response effect, in which **F**=gust energy ratio, **S**=size reduction factor, and D^t =total modal damping of the structure. An explanation of these factors follows;

1. The peak factor, **g**, (Figure 9.3a) is a function of the average fluctuation rate, μ, and the averaging period, **T**, which typically is a period between 5 minutes and 1 hour. The average fluctuation rate is

$$\mu = f_o\sqrt{\frac{SF/D^t}{B + SF/D^t}} \tag{9.10}$$

where f_o the natural frequency.

2. The roughness factor, **K**, is equal to
   0.08 for open terrain (Zone A)
   0.10 for suburban, or wooded terrain (Zone B)
   0.14 for concentrations of tall building (Zone C)

3. Exposure factor, C_e, is based on the mean speed profile and thus on surface roughness. For the three zones, the exposure factor is obtained from Figure 9.3b for the height of the building **H**. C_e relates to wind pressure rather than speed. Hence, the mean wind speed at the top of the building is, given by

$$\overline{V}_H = \overline{V}_{10}\sqrt{C_e} \tag{9.11}$$

where, $\overline{V}_{10}$=reference wind speed at the standard height of 10 meters, $\overline{V}_{10}$ can be obtained from meteorological stations.

4. The background turbulence factor, **B**, is obtained from Figure 9.3c as a function of height, **H**, and width, **W**, of the windward face of the structure.

5. The size reduction factor, **S**, (Figure 9.3d) depends on the reduced frequency $f_0 H / \overline{V}_H$ and the width to the height ratio **W/H**.

6. The gust energy factor, **F**, (Figure 9.3e) is a function of the wave number at resonance, $f_0 / \overline{V}_H$. Factor **F** represents the spectrum of the wind speed.

7. The total damping ratio, D^t depends on the structural damping D^s, damping due to energy dissipation in soil, **D**, and aerodynamic damping D^a, typical values for structural damping for structures on rigid foundations are:

Concrete structures: $D^s = 0.01\text{-}0.02$

Steel structures: $D^s = 0.005 - 0.01$

9.3 DESIGN WIND PRESSURE

The parameters given above yield the design wind pressure, **p**, which produces displacement $\overline{u}_{max}$ if applied as a static load. This design pressure is

$$p = qC_e\, G\, C_p \tag{9.12}$$

where $q = 1/2\rho\overline{V}_{10}^2$ is the reference mean velocity pressure, ρ=air density normally equals to 1.24 kg/m^3, and C_p=average pressure coefficient, which depends on the shape of the structure and the flow pattern around it. For a typical building with a flat roof and a height greater than twice the width, the coefficients are given for the windward and leeward faces in Figure 9.3f together with the pressure distribution. Exposure factor, C_e, varies continuosly with elevation according to Figure 9.3b for pressures acting on the wind ward face of the structure; for the leeward face, C_e is constant and is evaluated at one-half the height of the building.

Then the design wind pressure, **p**, is used to produce the equivalent static wind forces F_i at each floor **i** (figure 9.4) as

$$F_i = p_i . W . \ell_i \qquad i=1,2.........N \tag{9.13}$$

where ℓ_i = the tributary height of floor **i** and **W** is the width of the building. These forces applied to the structure yield the maximum displacements.

The factors evaluated from Figure 9.3 can also be used to estimate the peak acceleration at the top of the building, which is an important design criterion. Assuming that the total peak displacement Δ, caused by the loads given by Equation 9.12, corresponds to peak acceleration $a = \Delta\omega_0^2$ as in harmonic motions and that this acceleration is only due to the resonant part of the response, i.e. background turbulence factor **B**=0, the peak acceleration becomes

$$a = 4\pi^2 f_0^2 \frac{g}{G}\sqrt{\frac{KSF}{C_e D^t}}.\Delta \tag{9.14}$$

when the maximum acceleration exceeds 1% of gravity or even less, the motion is usually perceptible.

Problem:

Evaluate the gust factor, design pressure and acceleration for a tall building to be built on the downtown area. Consider exposures A&C.

Height	H=100 m
Width	W= 24m
Depth	DD= 20 m
Natural frequency	f_o =0.2 cps
Damping ratio	D= 0.01
Exposure	A, C

The reference hourly wind pressure is $q = 0.48$ kN/m^2

Evaluate the maximum acceleration assuming that Δ was found to be 0.02 m for both exposures

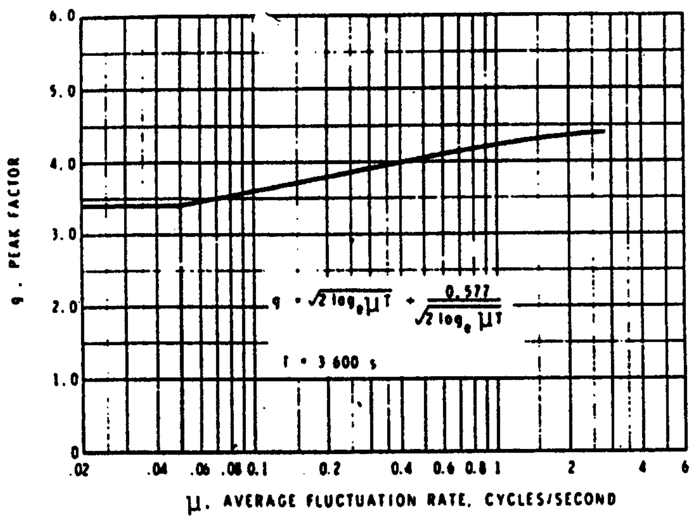


Figure 9.3a **Peak factor as a function of average fluctuation rate.**

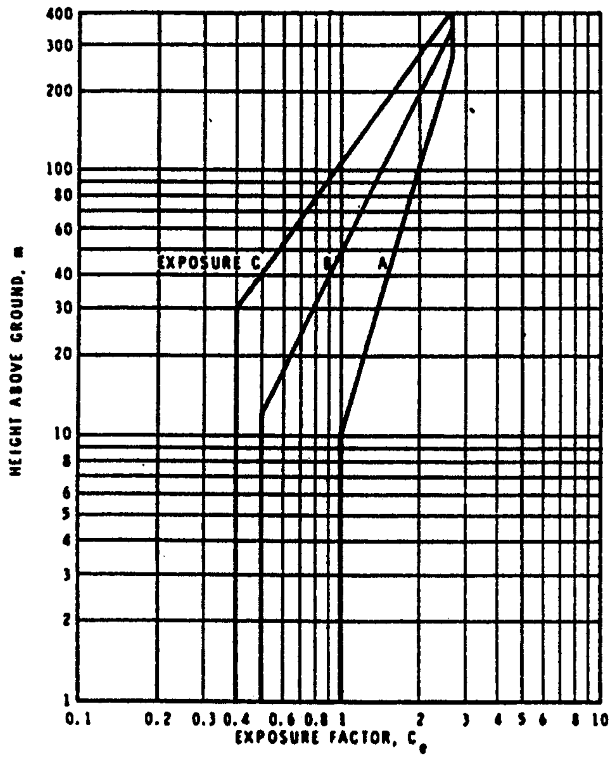


Figure 9.3b Exposure factor as a function of terrain roughness and height above ground.

$$C_e = \left(z / V_{10} \right)^{0.28} \qquad C_e \geq 1.0 \qquad \text{zone A}$$

$$C_e = 0.5\left(z / V_{10} \right)^{0.50} \qquad C_e \geq 0.5 \qquad \text{zone B}$$

$$C_e = 0.4\left(z / V_{10} \right)^{0.72} \qquad C_e \geq 0.5 \qquad \text{zone C}$$

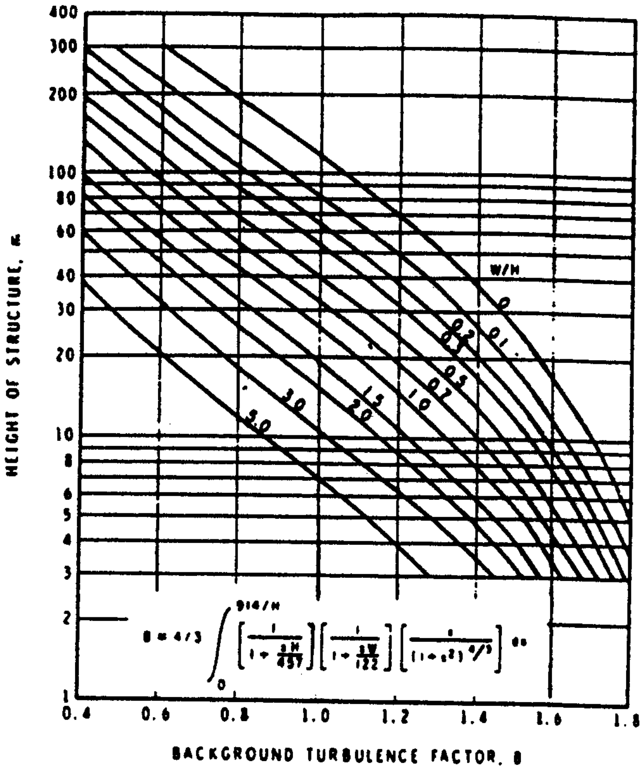


Figure 9.3c Background turbulence factor as a function of width and height of structures.

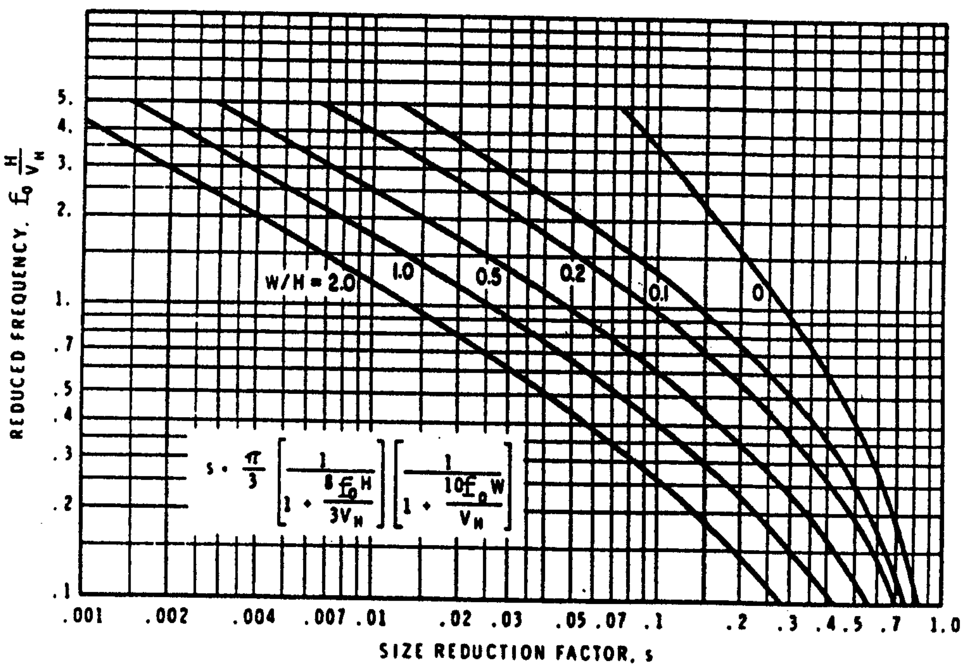


Figure 9.3d Size reduction factor as a function of width, height and reduced frequency of structure.

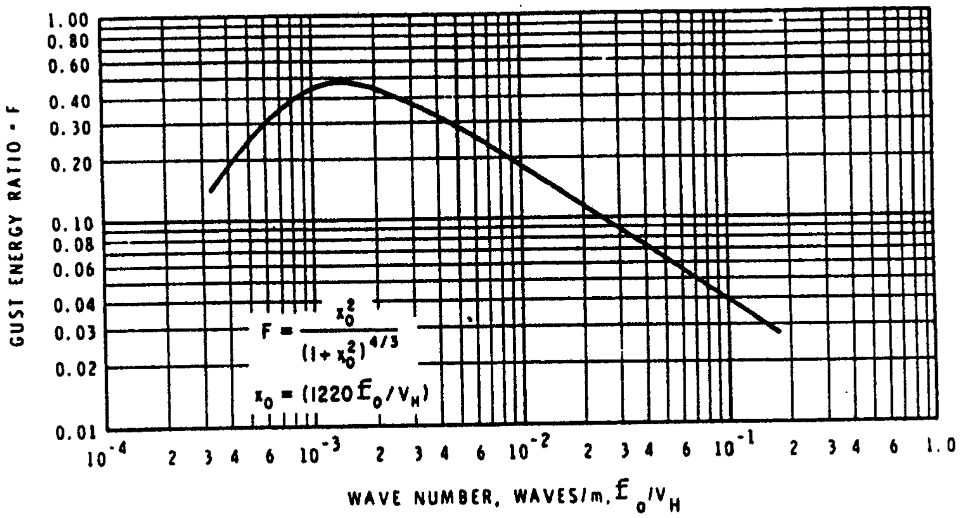


Figure 9.3e **Gust Energy Ratio as a function of wave number.**

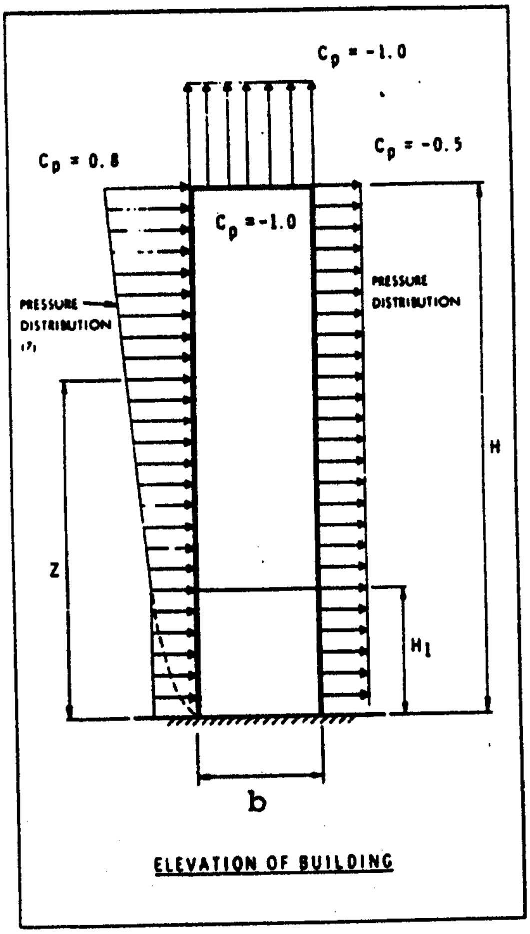


Figure 9.3f **Flat roof buildings of height greater than twice the width.**

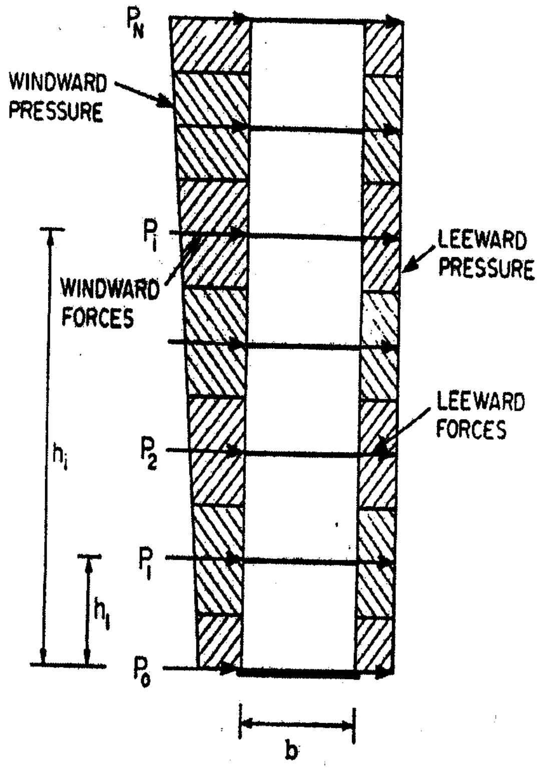


Figure 9.4 **Winward and Leeward forces due to wind pressure.**

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